10 Infinite Galois theory

10.1 Topological groups

Definition 10.1.1: A *topological group* is a group G endowed with a topology, such that the following maps are continuous:

$$G \times G \to G, \quad (g,h) \mapsto gh,$$
$$G \to G, \quad g \mapsto g^{-1}.$$

Example 10.1.2: Every group with the discrete topology is a topological group.

Remark 10.1.3: Some authors require that the topology is also Hausdorff.

Note: finite & Hausdorff $\Rightarrow$ discrete.

G
$\shortparallel$

Example 10.1.4: Let K be a field with an absolute value $|\ |$ and endow $\mathrm{GL}_n(K)$ with the topology induced by the product topology on $\mathrm{Mat}_{n\times n}(K) \cong K^{n^2}$.

$\shortparallel$
M

$M \times M \to M$, $(A,B) \mapsto A\cdot B$ continuous,

$G \to G$, $A \mapsto A^{-1} = \frac{1}{\det(A)} \cdot A^{\#}$

Proposition 10.1.5: Every subgroup of a topological group becomes a topological group with the induced topology.

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ \cup & & \cup \\ H \times H & \dashrightarrow & H \end{array} \qquad \begin{array}{ccc} G & \longrightarrow & G \\ \cup & & \cup \\ H & \longrightarrow & H \end{array}$$

Proposition 10.1.6: Every (finite or infinite) product of topological groups, endowed with the product topology, is a topological group.

$$\begin{tikzcd} \times_i G_i \times \times_i G_i \arrow[r] \arrow[d, phantom, "\cong"] & \times_{i\in I} G_i \\ \times_i (G_i \times G_i) \arrow[ru] & \end{tikzcd}$$

...

Proposition 10.1.7: For every topological group G and any $g \in G$ the maps $G \to G$, $x \mapsto gx$ and $x \mapsto xg$ and $x \mapsto {}^g x$ are homeomorphisms.

$x \mapsto g^{-1}x$

They are continuous, and so are their inverses

$x \mapsto xg^{-1}$

$x \mapsto {}^{g^{-1}}x$

Proposition 10.1.8: Every open subgroup of a topological group is closed.

Proof: $H < G$ open $\Rightarrow G = \dot{\bigcup}_{g \in R} \underbrace{gH}_{\text{open}}$ $\Rightarrow H = G \setminus \Big(\underbrace{\bigcup_{\substack{g\in R \\ g \notin H}} \underbrace{gH}_{\text{open}}}_{\text{open}} \Big)$ qed

R: sys of rep's

Definition 10.1.9: A *topological isomorphism* between topological groups is a group isomorphism which is also a homeomorphism.

Prop.: Every closed subgroup of finite index is open.

Proof: $H = G \setminus \Big(\bigcup_{\substack{g\in R \\ g \notin H}} \underbrace{gH}_{\text{closed}} \Big) =$ open. qed.

10.2 Profinite groups

Definition 10.2.1: A *profinite group* is a topological group that is topologically isomorphic to a closed subgroup of a (possibly infinite) product of discrete finite groups.

Proposition 10.2.2: For every profinite group G we have:

(a) G ist compact und Hausdorff.

(b) Every open subgroup has finite index.

(c) The open normal subgroups form a neighborhood base of the identity element.

(c) Neighborhood base

$$\left\{ G \cap \underbrace{\bigtimes_{i\in I} \begin{cases} G_i & i \notin J \\ \{1\} & i \in J \end{cases}}_{\text{open normal subgroup of } \bigtimes_i G_i} \;\middle|\; J \subset I \text{ finite} \right\}$$

Proof: (a) $\boxed{G < \bigtimes_{i\in I} G_i \quad \text{closed}, \; G_i \text{ finite discrete}}$ $\Rightarrow$ all G_i compact Hausdorff.

Tychonoff $\Rightarrow$ $\bigtimes_{i\in I} G_i$ compact $=$ Hausdorff ✓

qed.

(b) $H < G$ open $\Rightarrow$ $G = \dot{\bigcup}_{g\in R} \underbrace{gH}_{\text{open}}$ $\xRightarrow{\text{compact}}$ R finite. ✓

Example 10.2.3: The topology induced by the p-adic metric on $\mathbb{Z}_p$ is the same as that induced by the product topology on $\bigtimes_{n\geqslant 0} \mathbb{Z}/p^n\mathbb{Z}$. Thus the additive group $(\mathbb{Z}_p, +)$ and the group of units $(\mathbb{Z}_p^\times, \cdot)$ are profinite groups.

$\{p^n \mathbb{Z}_p \mid n \geq 0\}$ nbhd base of 0

Proposition 10.2.4: Every closed subgroup of a profinite group is a profinite group with the induced topology.

Proposition 10.2.5: Every factor group of a profinite group by a closed normal subgroup is a profinite group with the induced topology.

Proof: Take $N \triangleleft G$ closed.

For any $M \triangleleft G$ open $\Rightarrow NM \triangleleft G$ open $\Rightarrow G/NM$ finite

$\varphi : G \longrightarrow \bigtimes_M G/NM, \; g \longmapsto (gNM)_M$

Homo, continuous in each factor $\Rightarrow$ continuous.

$\ker(\varphi) = \bigcap_M NM \supset N$.

$\forall g \in (\bigcap_M NM) \setminus N$: N closed $\Rightarrow G \setminus N$ open

$\Rightarrow \exists M \triangleleft G$ open: $gM \subset G \setminus N$

$\Rightarrow gM \cap N = \emptyset \Rightarrow g \notin NM$.

$\Rightarrow$ Contradiction.

So $\ker(\varphi) = N$

$\Rightarrow \varphi$ factors through an injection $G/N \hookrightarrow \bigtimes_M G/NM$.

$\left.\begin{array}{l} G \text{ compact} \\ \bigtimes_M G/NM \text{ Hausdorff} \\ \varphi \text{ continuous} \end{array}\right\} \Rightarrow \operatorname{im}(\varphi) = \text{compact}$

$\Rightarrow$ closed.

qed.

Definition 10.2.6: The *profinite completion* of a group G is the profinite group

$$G \longrightarrow \varprojlim_N G/N \ := \ \Big\{(g_N N)_N \in \bigtimes_N G/N \ \Big| \ \forall\, N' \subset N \subset G \colon g_{N'}N = g_N N\Big\}, \quad =: \hat{G}$$

$$g \mapsto (gN)_N$$

where the product extends over all normal subgroups $N \lhd G$ of finite index.

Example 10.2.7: The profinite completion $\hat{\mathbb{Z}}$ of the group $\mathbb{Z}$ is isomorphic to $\prod_p \mathbb{Z}_p$.

subgroups of finite index $= n\mathbb{Z}$ for all $n \geq 1$ $\;|\;$ $n = \prod_p p^{\nu_p} \Rightarrow \mathbb{Z}/n\mathbb{Z} \cong \bigtimes_p \mathbb{Z}/p^{\nu_p}\mathbb{Z}$

$$\Rightarrow \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \varprojlim_{(\nu_p)_p} \bigtimes_p \mathbb{Z}/p^{\nu_p}\mathbb{Z} = \bigtimes_p \varprojlim_\nu \mathbb{Z}/p^\nu\mathbb{Z} = \bigtimes_p \mathbb{Z}_p .$$

Example: $G = \mathbb{Q} \Longrightarrow \hat{G} = 1$

10.3 Infinite Galois theory

Consider a galois extension of fields L/K which may or may not be finite.

Proposition 10.3.1: There is a natural injective group homomorphism

$$\mathrm{Aut}_K(L) =: \mathrm{Gal}(L/K) \to \bigtimes_{K'} \mathrm{Gal}(K'/K), \quad \gamma \mapsto (\gamma|_{K'})_{K'},$$

where the product extends over all intermediate fields K' that are finite and galois over K. Its image is the closed subgroup

$$\varprojlim_{K'} \mathrm{Gal}(K'/K) \;\; := \;\; \big\{(\gamma_{K'})_{K'} \;\big|\; \forall\, K''/K'/K\colon \gamma_{K''}|_{K'} = \gamma_{K'}\big\}.$$

This turns $\Gamma := \mathrm{Gal}(L/K)$ into a profinite group.

$$\begin{tikzcd} \gamma \circlearrowright L \arrow[d, dash] \\ \gamma_{K''} \circlearrowright K'' \arrow[d, dash] \\ \gamma_{K'} \circlearrowright K' \arrow[d, dash] \\ K \end{tikzcd}$$

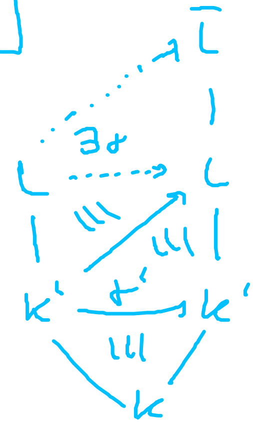
$$\begin{tikzcd} & K'' := K^+K' \arrow[dl, dash] \arrow[dr, dash] & \\ K^+ & & K' \end{tikzcd}$$

The subgroups $\{\gamma \in \Gamma \mid \gamma|_{K'} = \mathrm{id}\}$ for all $K' \subset L$ with K'/K finite galois form a neighborhood basis of $1 \in \Gamma$.

Theorem 10.3.2: *(Main Theorem of Galois theory)* There are natural mutually inverse bijections

$$\begin{array}{ccc} \{\text{Intermediate fields of } L/K\} & \overset{\sim}{\rightleftarrows} & \{\text{closed subgroups of } \Gamma\} \\ K' & \longmapsto & \mathrm{Gal}(L/K') \\ L^{\Gamma'} & \longleftarrow\!\shortmid & \Gamma' \end{array}$$

Here the open subgroups of Γ correspond to the subfields of finite degree over K.



Pf: ① $\forall K'$: $\mathrm{Gal}(L/K') \subset \mathrm{Gal}(L/K)$ closed ✓

② $\forall \Gamma'$: $L^{\Gamma'}$ is an intermediate field.

③ $\forall K'$: set $\Gamma' := \mathrm{Gal}(L/K')$. Claim: $L^{\Gamma'} = K'$.

Proof: WLOG: $K = K'$. Then $L^{\Gamma} \supset K$. Take $x \in L^{\Gamma} \setminus K$.

Take $K \subset K' \subset L$ with K'/K finite galois with $x \in K' \setminus K$

Then $\exists\, \gamma' \in \mathrm{Gal}(K'/K)$ with $\gamma'(x) \neq x$.

Extend γ' to $\gamma \in \mathrm{Gal}(L/K)$.

This contradicts $\gamma x \neq x \Rightarrow x \notin L^{\Gamma}$ good.

④ $\forall\, \Gamma' < \Gamma$ closed. Let $K' := L^{\Gamma'}$. Claim: $\Gamma' = \mathrm{Gal}(L/K')$.

Proof: "$\subset$" clear. Take $\sigma \in \mathrm{Gal}(L/K')$.

Take any $K \subset M \subset L$ with M/K finite galois.

$\Rightarrow MK'/K'$ finite galois.

M is Γ'-invariant $\Rightarrow MK'$ is Γ'-invariant (as a whole)

$(\Pi K')^{\Gamma'} \subset L^{\Gamma'} = K' \Rightarrow (\Pi K')^{\Gamma'} = K'$

$\Rightarrow \quad \Gamma' \twoheadrightarrow \mathrm{Gal}(\Pi K'/K')$.

$\Rightarrow \quad \exists\, \gamma' \in \Gamma' : \gamma'|_{\Pi K'} = \sigma|_{\Pi K'}$

$\Rightarrow \quad \gamma' \equiv \sigma$ mod $\mathrm{Gal}(L/\Pi K')$

Λ

$\Rightarrow \quad \gamma' \equiv \sigma$ mod $\mathrm{Gal}(L/\Pi)$ ← vary Π ⇒ find arbitrary nbhds of $1 \in \Gamma$.

$\Downarrow$

$\underline{\Gamma' \text{ closed}} \Longrightarrow \sigma \in \Gamma'$.

(5) K'/K finite ⇒ let K'' be its galois closure ⇒ $\mathrm{Gal}(L/K'') < \Gamma$ open

Λ

⇒ $\mathrm{Gal}(L/K')$ open.

(6) $\Gamma' < \Gamma$ open ⇒ $\exists\, K''/K$ finite galois: $\Gamma' > \mathrm{Gal}(L/K'') =: \Gamma''$

(3) ⇒ $L^{\Gamma''} = K''$ = finite over K

$\cup$

⇒ $L^{\Gamma'}$ finite over K.

qed