Reminder: We fix a nonarchimedean local field K with normalized valuation v and valuation ring $\mathcal{O}$ and finite residue field $k = \mathcal{O}/\mathfrak{m}$ of characteristic p.

Proposition 11.1.1: The reduction homomorphism $\mathcal{O}^\times \to k^\times$ is surjective and has a unique splitting, that is, a homomorphism $k^\times \to \mathcal{O}^\times$, $\alpha \mapsto \tilde{\alpha}$, such that $\tilde{\alpha} + \mathfrak{m} = \alpha$.

$q = |k|$ $\qquad X^{q-1} - 1$

$$\tilde{\alpha} = \lim_{n\to\infty} a^{q^n} \text{ for any } a \mapsto \alpha.$$

$k[t]$

Definition 11.1.3: A generator of the ideal $\mathfrak{m}$ is called a *uniformizer of K*.

Proposition 11.1.4: In the case $\mathrm{char}(K) = p$ we have:

(a) The ring homomorphism $\mathcal{O} \twoheadrightarrow k$ has a unique splitting, that is, a uniquely ring homomorphism $k \to \mathcal{O}$, $\alpha \mapsto \tilde{\alpha}$, such that $\tilde{\alpha} + \mathfrak{m} = \alpha$.

(b) For any uniformizer u of K there is a natural isomorphism $k((u)) \cong K$.

Proof: (a) The map must be that of 11.1.1 extended by $0 \mapsto 0$. That's multiplicative and $1 \mapsto 1$. Additive: Take $\alpha, \beta \in k^\times$ with lifts $a, b \in \mathcal{O}$.

$\widetilde{\alpha+\beta} = \tilde{\alpha} + \tilde{\beta}$ clear if $\alpha = 0$ or $\beta = 0$.

$$\Rightarrow \tilde{\alpha} + \tilde{\beta} = \lim_{n\to\infty} a^{q^n} + \lim_{n\to\infty} b^{q^n} = \lim_{n\to\infty} (a^{q^n} + b^{q^n})$$

$$= \lim_{n\to\infty} (a+b)^{q^n} = \begin{cases} \widetilde{\alpha+\beta} & \text{if } \alpha+\beta \neq 0 \\ 0 & \text{if } \alpha+\beta = 0 \end{cases}$$

$$= \widetilde{\alpha+\beta}$$

qed.

(b) $k[[u]] \xrightarrow{\sim} \mathcal{O}$, $\sum_i \alpha_i u^i \mapsto \sum_i \tilde{\alpha}_i u^i$.

$\Rightarrow k((u)) = k[[u]][\frac{1}{u}] \xrightarrow{\sim} K$.

qed.

Remark 11.1.5: One might think that this makes the theory of local fields of positive characteristic boring. But it does not tell us anything about the galois theory of such fields, which is as intricate as the galois theory of p-adic fields.

Proposition 11.1.6: (a) The group μ_K of roots of unity in $K^\times$ is finite.

(b) If K is an extension of degree n of $\mathbb{Q}_p$, there is an uncanonical isomorphism

$$K^\times \;\cong\; \mathbb{Z} \times \mu_K \times \mathbb{Z}_p^n.$$

(c) If K has characteristic p, there is an uncanonical isomorphism

$$K^\times \;\cong\; \mathbb{Z} \times k^\times \times \mathbb{Z}_p^{\mathbb{N}}.$$

Proof: u uniformizer, i.e., $v(u)=1$ $\Longrightarrow$ $\mathbb{Z}\times\mathcal{O}^\times \xrightarrow{\sim} K^\times$, $(n,x)\mapsto u^n\cdot x$.

$$\boxed{\mathcal{O}^\times \cong \underbrace{k^\times}_{\text{finite of order } |k^\times|} \times (1+\mathfrak{m},\cdot)}$$

(b) Recall: $\mathbb{Z}_p^n \cong p^2\cdot\mathcal{O} \underset{\log}{\overset{\exp}{\rightleftarrows}} 1+p^2\cdot\mathcal{O}$

$\cap$ finite index $[\mathfrak{m} : p^2\mathcal{O}]$ = power of p.

$1+\mathfrak{m}$ $\Longleftarrow$ $\mathbb{Z}_p$-module, fin. gen.

$\Longrightarrow \mathcal{O}^\times \cong \overbrace{k^\times \times (\text{finite})}^{\text{tors}} \times \mathbb{Z}_p^n$

$\Longrightarrow \mathbb{Z}_p^m \times (\text{finite}) \Longrightarrow m=n.$

(c) Let $\alpha_1, \dots, \alpha_r$ be a basis of k over $\mathbb{F}_p$.

$\boxed{K = k((u))}$ for all $1 \leq i \leq r$ and $p \nmid j > 0$ set $x_{ij} := 1 + \alpha_i u^j$ $\equiv 1 \mod m^j$.

Claim:

$$\boxed{\begin{aligned} \bigtimes_{i=1}^{r} \bigtimes_{p \nmid j > 0} \mathbb{Z}_p &\xrightarrow{\sim} (1+m, \cdot), \\ (n_{ij}) &\longmapsto \prod_{i=1}^{r} \prod_{p \nmid j > 0} x_{ij}^{n_{ij}} \end{aligned}}$$

well defined

continuous

homom.

$$(1+u^j)^p = 1 + u^{pj}$$

qed

11.2 Unramified extensions

i.e. $e=1$, $\iff f:=[\ell/k]=[L/K]$.
$\iff$ Inertia group $=1$

Proposition 11.2.1: For any integer $n \geqslant 1$ there exists an unramified extension L/K of degree n. It is unique up to isomorphism over K and galois over K. Its residue field ℓ is an extension of degree n of k, and $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(\ell/k)$ is cyclic of order n.

Proof: If L/K is unramified of degree n, then $|\ell| = q^n$ for $q=|k| \xrightarrow{11.1.1} \mu_{q^n-1} \cong \ell^\times \hookrightarrow L^\times$.

Also $K \subset K(\mu_{q^n-1}) =: L' \subset L$, unram. with residue field $\ell' \subset \ell \Rightarrow \mu_{q^n-1} \subset \ell'^\times \Rightarrow \ell' = \ell$

$\Rightarrow L' = L$. $\Rightarrow L = K(\mu_{q^n-1}) \Rightarrow$ uniqueness.

Remark: For any $n \geq 1$ then claim $L := K(\mu_{q^n-1})/K$ is unramified of degree n.

Let ℓ be its residue field $\Rightarrow \mu_{q^n-1} \subset \ell^\times \Rightarrow [\ell/k] \geq n$.

$X^{q^n-1}-1$ is separable over $\mathbb{F}_p$, take $\gamma \in \bar{k}$ a primitive $(q^n-1)^{st}$ root of unity.

$\Rightarrow |k(\gamma)| = q^n \Rightarrow [k(\gamma)/k] = n \Rightarrow$ the min. pol. $f \in k[X]$ of γ over k has degree n.

This is a simple factor of $X^{q^n-1}-1$.

Hensel's lemma $\Rightarrow \exists!$ factorization $X^{q^n-1}-1 = \tilde{f}\cdot\tilde{g}$ with $\tilde{f} \equiv f$ and so

A root $\tilde{\gamma}$ of $\tilde{f}$ is a primitive $(q^n-1)^{st}$ root of unity. $\Rightarrow [K(\tilde{\gamma})/K] = n$ and $K(\tilde{\gamma}) = K(\mu_{q^n-1})$

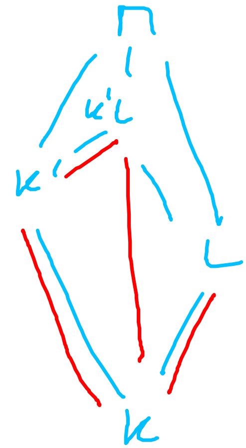
$\tilde{f}$ and so separable $\Rightarrow$ unramified. q.e.d.

Proposition 11.2.2: Consider a finite extension M/K with intermediate fields K', L.

(a) M/K is unramified if and only if M/L and L/K are unramified.

(b) If L/K is unramified, then so is LK'/K'.

(c) If L/K and K'/K is unramified, then so is LK'/K.



Proof: (a) 9.4.2(a):

$$\begin{cases} e_{L/K} = [v(L^\times) : v(K^\times)] \\ e_{M/L} = [v(M^\times) : v(L^\times)] \\ e_{M/K} = [v(M^\times) : v(K^\times)] \overset{\text{Lagrange}}{=} e_{M/L} \cdot e_{L/K} \end{cases}$$

$$\underset{\overset{\shortparallel}{1}}{e_{M/K}} \quad \Leftrightarrow \quad \underset{\overset{\shortparallel}{1}}{e_{M/L}} \cdot \underset{\overset{\shortparallel}{1}}{e_{L/K}}$$

(b) L/K unramified $\Rightarrow L = K(\mu_m)$ for some $p \nmid m$

$\Rightarrow K'L = K'(\mu_m) \Rightarrow K'L/K'$ unramified.

(c) $\Leftarrow$ (a) & (b)

qed.

Definition 11.2.3: An algebraic extension L/K is called *unramified* if it is a union of unramified finite extensions of K.

Proposition 11.2.4: (a) There exists a maximal unramified extension K^{nr} and it is unique up to isomorphism over K, though the isomorphism is not unique.

(b) The extension K^{nr}/K is galois. The residue field $\bar{k}$ of $\mathcal{O}_{K^{\mathrm{nr}}}$ is an algebraic closure of k and there are canonical isomorphisms

$$\mathrm{Gal}(K^{\mathrm{nr}}/K) \;\cong\; \mathrm{Gal}(\bar{k}/k) \;\cong\; \hat{\mathbb{Z}}.$$

Proof: $K^{nr} = \bigcup_{n \geq 1} K(\mu_{q^n - 1})$

$$\mathrm{Gal}(K^{nr}/K) = \varprojlim \mathrm{Gal}(K(\mu_{q^n-1})/K) = \varprojlim (\mathbb{Z}/n\mathbb{Z}) = \hat{\mathbb{Z}}.$$

End

11.3 Tame extensions

$e_{L/K}$

Definition 11.3.1: A finite extension L/K is called *tame* if its ramification index is not divisible by p.

Fact: unramified $\Rightarrow$ tame.

Proposition 11.3.2: (a) Any extension of the form $K(\sqrt[e]{a})/K$ for $p \nmid e \geqslant 1$ and $a \in K$ is tame.

(b) If in addition $v(a)$ is coprime to e, the extension is totally ramified of degree e.

Proof (a) Take a uniformizer π of K, write $a = \pi^r u$ for a unit $u \in \mathcal{O}^\times$ and $r \in \mathbb{Z}$.

Let $\tilde{\pi}$ be a root of $X^e - \pi$ $\Rightarrow$ $v(\tilde{\pi}) = \frac{1}{e} \cdot v(\pi)$

$\Rightarrow$ $K(\tilde{\pi})/K$ is totally ramified of degree e. $\Rightarrow$ tame

Let α be a root of $X^e - a$ $\Rightarrow$ $\left(\frac{\alpha}{\tilde{\pi}^r}\right)^e = \frac{a}{(\tilde{\pi}^e)^r} = \frac{a}{\pi^r} = u$

$\Rightarrow$ $\frac{\alpha}{\tilde{\pi}^r}$ is a root of $X^e - u$ $\leftarrow$ is separable mod $\mathfrak{m}$.

$\Rightarrow$ $K\left(\frac{\alpha}{\tilde{\pi}^r}\right)/K$ unramified $\Rightarrow$ $K(\alpha, \tilde{\pi})/K(\tilde{\pi})$ unramified

$\Rightarrow$ $K(\alpha, \tilde{\pi})/K$ tame $\Rightarrow$ $K(\alpha)/K$ tame.

(b) Friday!