

Reminder:

Definition 11.3.1: A finite extension L/K is called *tame* if its ramification index is not divisible by p .

Proposition 11.3.2: (a) Any extension of the form $K(\sqrt[e]{a})/K$ for $p \nmid e \geq 1$ and $a \in K$ is tame.

(b) If in addition $v(a)$ is coprime to e , the extension is totally ramified of degree e .

Proof of (b) : π uniformiser of K
 $a = \pi^r u$ for $r \in \mathbb{Z}$ and $u \in \mathcal{O}^\times$
Then $(e, r) = 1$
 $\Rightarrow \exists i, j \in \mathbb{Z} : ir - je = 1$
 $\alpha := \sqrt[e]{a} \Rightarrow v\left(\frac{\alpha^i}{\pi^j}\right) = \frac{1}{e} \cdot v\left(\frac{\alpha^{ie}}{\pi^{je}}\right) = \frac{1}{e} \cdot v\left(\frac{a^i}{u^{je}}\right) = \frac{1}{e} (ir - je) = \frac{1}{e}$
 \Rightarrow ramification index is divisible by e
Also $[K(\alpha)/K] \leq e$ } $\Rightarrow [K(\alpha)/K] = e$
totally ramified. qed.

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Proposition 11.3.3: Any finite extension L/K that is tame and totally ramified of degree e has the form $L = K(\sqrt[e]{\pi})$ for a uniformizer $\pi \in K$.

Proof, let $\tilde{\pi} \in L$ with $v(\tilde{\pi}) = \frac{1}{e}$, i.e., a uniformizer of L .

Let $f \in K[X]$ be its min. pol. over K .

$\tilde{\pi}$ is a root of $f \Rightarrow f \in \mathcal{O}[X]$.

The intermediate field $K(\tilde{\pi})$ has norm. degree $\geq e \Rightarrow K(\tilde{\pi}) = L \Rightarrow \deg(f) = e$.

$f(X) = X^e + \sum_{i=0}^{e-1} a_i X^i \Rightarrow$ all $a_i \in \mathcal{M}$, i.e., $v(a_i) \geq 1$.

$$\tilde{\pi}^e = - \sum_{i=0}^{e-1} a_i \tilde{\pi}^i$$

$$v = e \cdot \frac{1}{e} = 1$$

$$v \geq \frac{v(a_i) + i \cdot \frac{1}{e}}{\geq 1} > 1 \text{ if } i > 0$$

$$= v(a_0) \text{ if } i = 0$$

$\Rightarrow v(a_0) = 1$

So f is an Eisenstein poly. mod.

$\Rightarrow \pi := -a_0$ is a uniformizer of K with $\tilde{\pi}^e \equiv \pi \pmod{\mathcal{M}^2}$

i.e. $|\tilde{\pi}^e - \pi| < |\pi|$

Take $g(X) := X^e - \pi$. Let $\alpha_1, \dots, \alpha_e$ be its roots in \bar{L} .

$\Rightarrow v(\alpha_i) = \frac{1}{e} \Rightarrow |\alpha_i| = |\tilde{\pi}|$

$$\left. \begin{aligned} v(g(\tilde{\alpha})) &= v(\tilde{\alpha}^e - \alpha) > 1 \\ \sum_i v(\tilde{\alpha} - \alpha_i) \end{aligned} \right\} \Rightarrow \exists i, \text{ only } i=1 : v(\tilde{\alpha} - \alpha_1) > \frac{1}{e} .$$

$$\Rightarrow |\tilde{\alpha} - \alpha_1| < |\alpha_1|$$

$$\left. \begin{aligned} v(g'(\alpha_1)) &= v(e \cdot \alpha_1^{e-1}) = (e-1) \cdot v(\alpha_1) = \frac{e-1}{e} \\ v(\alpha_1 - \alpha_2) \sim (\alpha_1 - \alpha_e) &= \sum_{i=2}^e \underbrace{v(\alpha_1 - \alpha_i)}_{\geq 1/e} \end{aligned} \right\} \Rightarrow \forall i > 1 : v(\alpha_1 - \alpha_i) = \frac{1}{e}$$

$$\forall i > 1 : |\alpha_1 - \alpha_i| = |\alpha_1|$$

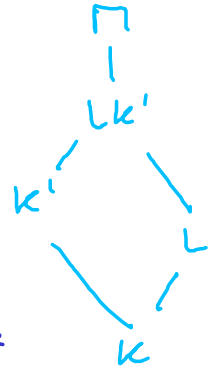
$$|\tilde{\alpha} - \alpha_1| < |\alpha_1|$$

$$\text{Krasner's lemma} \Rightarrow \alpha_i \in L . \Rightarrow L = K(\alpha_1)$$

qed

Proposition 11.3.4: Consider a finite extension M/K with intermediate fields K', L .

- (a) M/K is tame if and only if M/L and L/K are tame.
- (b) If L/K is tame, then so is LK'/K' .
- (c) If L/K and K'/K is tame, then so is LK'/K .



Pf: (a) $e_{M/K} = e_{L/K} \cdot e_{M/L}$

(b) L/K tame $\Rightarrow \exists ! N: N/K$ unramified
 L/N totally ramified $\Rightarrow L = N(\sqrt[p]{a})$ with $p \nmid e$

$\Rightarrow NK'/K'$ unramified
 $LK' = NK'(\sqrt[p]{a})$ tame
 $\Leftrightarrow LK'/K'$ tame.

(c) \Leftarrow (a) \wedge (b)

qed

Definition 11.3.5: An algebraic extension L/K is called tame if it is a union of tame finite extensions of K .

Proposition 11.3.6: (a) There exists a maximal tame extension K^{tr} and it is unique up to isomorphism over K , though the isomorphism is not unique.

(b) The extension K^{tr}/K is galois and contains a maximal unramified extension K^{nr} .

Proof: Work inside $\bar{K} \Rightarrow$ take the union of all tame finite extensions of K .

tame \Rightarrow separable
 K^{tr}/K unram \Rightarrow galois.

qed

Proposition 11.3.7: (a) The Galois group of $K^{\text{tr}}/K^{\text{nr}}$ is naturally isomorphic to

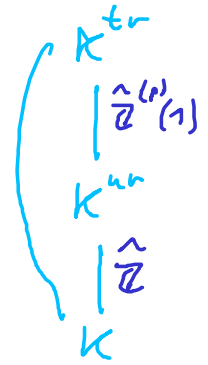
$$\hat{\mathbb{Z}}^{(p)}(1) := \varprojlim_n \mu_n(\bar{k}) := \left\{ (\zeta_n)_n \in \prod_n \mu_n(\bar{k}) \mid \forall n|n': \zeta_{n'}^{n'/n} = \zeta_n \right\},$$

where the product extends over all integers $p \nmid n \geq 1$.

(d) The Galois group of K^{tr}/K is isomorphic to the semidirect product

$$\hat{\mathbb{Z}} \ltimes \hat{\mathbb{Z}}^{(p)}(1),$$

where $1 \in \hat{\mathbb{Z}}$ acts on $\hat{\mathbb{Z}}^{(p)}(1)$ by the map $x \mapsto x^{|k|}$.



$$\begin{array}{ccc}
 \mu_n(\bar{k}) & \xrightarrow{\gamma \mapsto \gamma^{1/n}} & \mu_n(\bar{k}) \\
 \parallel & & \parallel \\
 \mathbb{Z}/n\mathbb{Z} & \xrightarrow{(\cdot)^2} & \mathbb{Z}/n\mathbb{Z}
 \end{array}
 \quad \Rightarrow \quad
 \begin{array}{ccc}
 \varprojlim_n \mu_n(\bar{k}) & = & \hat{\mathbb{Z}}^{(p)}(1) \\
 \parallel & & \\
 \varprojlim_n \mathbb{Z}/n\mathbb{Z} & = & \prod_{p \nmid n} \mathbb{Z} \leftarrow \hat{\mathbb{Z}}
 \end{array}$$

Remark 11.3.8: This inverse limit is uncanonically isomorphic to the prime-to- p part

$$\hat{\mathbb{Z}}^{(p)} := \prod_{\ell \neq p} \mathbb{Z}_\ell$$

of the profinite completion $\hat{\mathbb{Z}}$ of \mathbb{Z} . The notation $\hat{\mathbb{Z}}^{(p)}(1)$ is chosen to indicate the nontrivial action of $\text{Gal}(K^{\text{nr}}/K)$.