Reminder:

Definition 11.3.1: A finite extension L/K is called *tame* if its ramification index is not divisible by p.

Proposition 11.3.2: (a) Any extension of the form $K(\sqrt[e]{a})/K$ for $p \nmid e \geqslant 1$ and $a \in K$ is tame.

(b) If in addition $v(a)$ is coprime to e, the extension is totally ramified of degree e.

Proof of (b): π uniformizer of K

$a = \pi^r u$ for $r \in \mathbb{Z}$ and $u \in \mathcal{O}^\times$

Then $(e, r) = 1$

$\Rightarrow \exists\, i, j \in \mathbb{Z} : \; ir - je = 1$

$$\alpha := \sqrt[e]{a} \Rightarrow v\left(\frac{\alpha^i}{\pi^j}\right) = \frac{1}{e} \cdot v\left(\frac{\alpha^{ie}}{\pi^{je}}\right) = \frac{1}{e} \cdot v\left(\frac{a^i}{\pi^{je}}\right) = \frac{1}{e}\left(ir - je\right) = \frac{1}{e}$$

$\Rightarrow$ ramification degree is divisible by e

Also $[K(\alpha)/K] \leq e$

$\Rightarrow$ $[K(\alpha)/K] = e$ totally ramified. qed.

$\Rightarrow p \nmid e$

Proposition 11.3.3: Any finite extension L/K that is tame and totally ramified of degree e has the form $L = K(\sqrt[e]{\pi})$ for a uniformizer $\pi \in K$.

Proof: Let $\tilde{\pi} \in L$ with $v(\tilde{\pi}) = \frac{1}{e}$, i.e., a uniformizer of L.

Let $f \in K[X]$ be its min. pol. over K.

$\tilde{\pi}$ integral over $\mathcal{O}$ $\Rightarrow f \in \mathcal{O}[X]$.

The intermediate field $K(\tilde{\pi})$ has ram. degree $\geq e$ $\Rightarrow K(\tilde{\pi}) = L \Rightarrow \deg(f) = e$.

$f(X) = X^e + \sum_{i=0}^{e-1} a_i X^i \Rightarrow$ all $a_i \in \mathfrak{m}$, i.e., $v(a_i) \geq 1$.

$$\underbrace{\tilde{\pi}^e}_{v = e \cdot \frac{1}{e} = 1} = -\sum_{i=0}^{e-1} \underbrace{a_i \tilde{\pi}^i}_{v \geq \underbrace{v(a_i)}_{\geq 1} + i \cdot \frac{1}{e} \begin{cases} > 1 & \text{if } i > 0 \\ = v(a_0) & \text{if } i = 0 \end{cases}} \quad \Rightarrow v(a_0) = 1$$

So f is an Eisenstein polynomial.

$\Rightarrow$ $\pi := -a_0$ is a uniformizer of K with $\tilde{\pi}^e \equiv \pi \mod \mathfrak{m}^2$

i.e. $|\tilde{\pi}^e - \pi| < |\pi|$

Take $\underline{g(X) := X^e - \pi}$. Let $\alpha_1, \ldots, \alpha_e$ be its roots in $\overline{L}$.

$\Rightarrow \forall i: \ v(\alpha_i) = \frac{1}{e} \Rightarrow |\alpha_i| = |\tilde{\pi}|$

$$\left.\begin{array}{l} v(g(\tilde{\pi})) = v(\tilde{\pi}^e - \pi) > 1 \\ \quad \shortparallel \\ \sum_i v(\tilde{\pi} - \alpha_i) \end{array}\right\} \Rightarrow \exists\, i, \text{ say } i=1: \ v(\tilde{\pi} - \alpha_1) > \frac{1}{e}.$$

$$\Rightarrow |\tilde{\pi} - \alpha_1| < |\alpha_1|$$

$$\left.\begin{array}{l} v(g'(\alpha_1)) = v(e \cdot \alpha_1^{e-1}) = (e-1)\cdot v(\alpha_1) = \frac{e-1}{e} \\ \quad \shortparallel \\ v((\alpha_1 - \alpha_2) \cdots (\alpha_1 - \alpha_e)) = \sum_{i=2}^{e} \underbrace{v(\alpha_1 - \alpha_i)}_{\geq 1/e} \end{array}\right\} \Rightarrow \forall i > 1: \ v(\alpha_1 - \alpha_i) = \frac{1}{e}$$

$\forall i > 1: \ |\alpha_1 - \alpha_i| = |\alpha_1|$

$\qquad\qquad |\tilde{\pi} - \alpha_1| < |\alpha_1|$

Krasner's lemma $\Rightarrow \alpha_1 \in L. \Rightarrow L = K(\alpha_1)$

qed

Proposition 11.3.4: Consider a finite extension M/K with intermediate fields K', L.

(a) M/K is tame if and only if M/L and L/K are tame.

(b) If L/K is tame, then so is LK'/K'.

(c) If L/K and K'/K is tame, then so is LK'/K.

$$\begin{tikzcd}[column sep=small]
& M \arrow[d, dash] & \\
& LK' \arrow[dl, dash] \arrow[dr, dash] & \\
K' \arrow[dr, dash] & & L \arrow[dl, dash] \\
& K &
\end{tikzcd}$$

Pf: (a) $e_{M/K} = e_{L/K} \cdot e_{M/L}$

(b) L/K tame $\Rightarrow \exists!\, N$: N/K unramified, L/N totally ramified & tame $\Rightarrow L = N(\sqrt[e]{a})$ with $p \nmid e$

$\Rightarrow NK'/K'$ unramified, $LK' = NK'(\sqrt[e]{a})$ tame $\overset{(a)}{\Rightarrow} LK'/K'$ tame.

(c) $\Leftarrow$ (a) $\wedge$ (b)

qed

Definition 11.3.5: An algebraic extension L/K is called *tame* if it is a union of tame finite extensions of K.

Proposition 11.3.6: (a) There exists a maximal tame extension K^{tr} and it is unique up to isomorphism over K, though the isomorphism is not unique.

(b) The extension K^{tr}/K is galois and contains a maximal unramified extension K^{nr}.

Proof: Work inside $\overline{K}$ $\Rightarrow$ take the union of all tame finite extensions of K.

$\left.\begin{array}{l} \text{tame} \Rightarrow \text{separable} \\ K^{\mathrm{tr}}/K \text{ normal} \end{array}\right\} \Rightarrow$ galois.

qed.

Proposition 11.3.7: (a) The galois group of $K^{\mathrm{tr}}/K^{\mathrm{nr}}$ is naturally isomorphic to

$$\hat{\mathbb{Z}}^{(p)}(1) \;:=\; \varprojlim_n \mu_n(\bar{k}) \;:=\; \Big\{(\zeta_n)_n \in \bigtimes_n \mu_n(\bar{k}) \;\Big|\; \forall\, n|n' \colon \zeta_{n'}^{n'/n} = \zeta_n\Big\},$$

where the product extends over all integers $p \nmid n \geqslant 1$.

(d) The galois group of K^{tr}/K is isomorphic to the semidirect product

$$\hat{\mathbb{Z}} \ltimes \hat{\mathbb{Z}}^{(p)}(1),$$

where $1 \in \hat{\mathbb{Z}}$ acts on $\hat{\mathbb{Z}}^{(p)}(1)$ by the map $x \mapsto x^{|k|}$.

$$\begin{array}{ccc} K^{tr} & & \\ \Big| & \hat{\mathbb{Z}}^{(p)}(1) & \\ K^{nr} & & \\ \Big| & \hat{\mathbb{Z}} & \\ K & & \end{array}$$

$n|n'$

$$\begin{array}{ccc} \mu_{n'}(\bar{k}) & \xrightarrow{\zeta \mapsto \zeta^{n'/n}} & \mu_n(\bar{k}) \\ \wr\Vert & /// & \wr\Vert \\ \mathbb{Z}/n'\mathbb{Z} & \xrightarrow{\text{(id)}} & \mathbb{Z}/n\mathbb{Z} \end{array} \quad \Rightarrow \quad \begin{array}{l} \varprojlim_n \mu_n(\bar{k}) = \hat{\mathbb{Z}}^{(p)}(1) \\ \quad \wr\Vert \\ \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \prod_{\ell \neq p} \mathbb{Z}_\ell \twoheadleftarrow \hat{\mathbb{Z}} \end{array}$$

Remark 11.3.8: This inverse limit is uncanonically isomorphic to the prime-to-p part

$$\hat{\mathbb{Z}}^{(p)} := \prod_{\ell \neq p} \mathbb{Z}_\ell$$

of the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$. The notation $\hat{\mathbb{Z}}^{(p)}(1)$ is chosen to indicate the nontrivial action of $\mathrm{Gal}(K^{\mathrm{nr}}/K)$.