Reminder:

We fix a nonarchimedean local field K with normalized valuation v and valuation ring $\mathcal{O}$ and finite residue field $k = \mathcal{O}/\mathfrak{m}$ of characteristic p.

Proposition 11.3.2: (a) Any extension of the form $K(\sqrt[e]{a}\,)/K$ for $p \nmid e \geqslant 1$ and $a \in K$ is tame.

(b) If in addition $v(a)$ is coprime to e, the extension is totally ramified of degree e.

Proposition 11.3.3: Any finite extension L/K that is tame and totally ramified of degree e has the form $L = K(\sqrt[e]{u}\,)$ for a uniformizer $u \in K$.

Proposition 11.3.6: (a) There exists a maximal tame extension K^{tr} and it is unique up to isomorphism over K, though the isomorphism is not unique.

(b) The extension K^{tr}/K is galois and contains a maximal unramified extension K^{nr}.

Proposition 11.3.7: (a) The galois group of $K^{\mathrm{tr}}/K^{\mathrm{nr}}$ is naturally isomorphic to

$$\hat{\mathbb{Z}}^{(p)}(1) \;:=\; \boxed{\varprojlim_n \mu_n(\bar{k})} \;:=\; \Big\{(\zeta_n)_n \in \bigtimes_n \mu_n(\bar{k}) \;\Big|\; \forall\, n|n' \colon \zeta_{n'}^{n'/n} = \zeta_n\Big\},$$

where the product extends over all integers $p \nmid n \geqslant 1$.

(b) The galois group of K^{tr}/K is isomorphic to the semidirect product

$$\hat{\mathbb{Z}} \ltimes \hat{\mathbb{Z}}^{(p)}(1),$$

where $1 \in \hat{\mathbb{Z}}$ acts on $\hat{\mathbb{Z}}^{(p)}(1)$ by the map $x \mapsto x^{|k|}$.

$$\begin{tikzcd} K^{tr} \arrow[d, dash] \\ K^{nr} \arrow[d, dash, "\hat{\mathbb{Z}} \ni 1 \leftrightarrow \mathrm{Frob}_{|k|}"] \\ K \end{tikzcd}$$

Proof: (a) $\forall\, L/K$ finite tame, $L \subset K^{tr}$: LK^{nr}/K^{nr} totally ramified of degree n.

$\Rightarrow LK^{nr} = K^{nr}(\sqrt[n]{\pi})$ for some uniformizer $\pi \in \mathcal{O}_{K^{nr}}$.

$\mathrm{Gal}(LK^{nr}/K^{nr}) \xrightarrow{\sim} \mu_n$, $\quad \sigma \longmapsto \dfrac{\sigma\sqrt[n]{\pi}}{\sqrt[n]{\pi}}$

Any other uniformizer is πu for $u \in \mathcal{O}_{K^{nr}}^\times$ and $K(\sqrt[n]{u})/K$ unramified. $X^n - u$ separable mod $\mathfrak{m}$

$\Downarrow$

this iso. depends only on n.

Take $\varprojlim_{p \nmid n}$ $\rightsquigarrow$ ✓

(b) $1 \to \hat{\mathbb{Z}}^{(p)}(1) \to \mathrm{Gal}(K^{tr}/K) \to \hat{\mathbb{Z}} \to 0 \Rightarrow \mathrm{Gal}(K^{tr}/K) \cong \hat{\mathbb{Z}} \ltimes \hat{\mathbb{Z}}^{(p)}(1)$

Choose any lift $\sigma \longmapsto 1 \in \hat{\mathbb{Z}}$, $\quad \sigma^n \longleftarrow n$

qed

$v_L|_K = e_{L/K} \cdot v_K$

11.4 The lower numbering filtration

Fix a finite galois extension L/K with galois group Γ and residue field $\ell = \mathcal{O}_L/\mathfrak{m}_L$. Let v_L denote the normalized valuation on L.

Definition 11.4.1: For every real number $s \geqslant -1$ we define the *s-th ramification group of L/K in the lower numbering* as

$$\boxed{\Gamma_s := \{\gamma \in \Gamma \mid \forall a \in \mathcal{O}_L : v_L({}^\gamma a - a) \geqslant s+1\}.}$$

$\iff \gamma$ acts trivially on $\mathcal{O}_L/\mathfrak{m}_L^{\lceil s\rceil+1}$

Proposition 11.4.2: (a) For every s we have $\Gamma_s = \Gamma_{\lceil s\rceil}$. ✓

(b) For every s we have $\Gamma_s \lhd \Gamma$. ✓ in fact: $\Gamma_s = \ker\left(\Gamma \to \mathrm{Aut}\left(\mathcal{O}_L/\mathfrak{m}_L^{\lceil s\rceil+1}\right)\right)$.

(c) We have $\Gamma_{-1} = \Gamma$. ✓ because $\mathfrak{m}_L^{-1+1} = \mathcal{O}_L$.

(d) The subgroup Γ_0 is the inertia group. ✓ because $\mathcal{O}_L/\mathfrak{m}_L = \ell$

(e) There exists s with $\Gamma_s = 1$. $\forall \gamma \in \Gamma\setminus\{1\} : \exists a \in \mathcal{O}_L : {}^\gamma a \neq a \Rightarrow v_L({}^\gamma a - a) < \infty$

$\Rightarrow \gamma \notin \Gamma_{v_L({}^\gamma a - a)+2}$

Also $\forall s' \geq s : \Gamma_{s'} < \Gamma_s$.

qed.

Proposition 11.4.3: (a) There is a natural isomorphism $\Gamma/\Gamma_0 \cong \mathrm{Gal}(\ell/k)$. ✓

(b) Any uniformizer u of L induces an injective homomorphism

$$\Gamma_0/\Gamma_1 \hookrightarrow \ell^\times, \quad [\gamma] \mapsto \left[\frac{{}^\gamma u}{u}\right].$$

independent of the choice of u.

(c) For any integer $s \geqslant 1$, any uniformizer u of L induces an injective homomorphism

$$\Gamma_s/\Gamma_{s+1} \hookrightarrow (\ell, +), \quad [\gamma] \mapsto \left[\frac{{}^\gamma u - u}{u^{s+1}}\right].$$

Proof: (a) ✓

(b) $v_L(u)=1 \Rightarrow v_L({}^\gamma u)=1 \Rightarrow v_L\left(\frac{{}^\gamma u}{u}\right)=0 \Rightarrow$ map well defined $\Gamma_0 \to \ell^\times$

$\gamma, \delta \in \Gamma_0 \Rightarrow \frac{{}^{\gamma\delta}u}{u} = {}^\gamma\Big(\underbrace{\frac{{}^\delta u}{u}}_{\in \mathcal{O}_L}\Big)\cdot\left(\frac{{}^\gamma u}{u}\right)$

$\overset{\gamma\in\Gamma_0}{\Rightarrow} {}^\gamma\left(\frac{{}^\delta u}{u}\right) \equiv \frac{{}^\delta u}{u}$ mod m

$\Rightarrow \left[\frac{{}^{\gamma\delta}u}{u}\right] = \left[\frac{{}^\delta u}{u}\right]\cdot\left[\frac{{}^\gamma u}{u}\right] \Rightarrow$ map is a homo.

$\Gamma_0 \ni \gamma \in \ker \Leftrightarrow \frac{{}^\gamma u}{u} \equiv 1$ mod $m_L \Leftrightarrow {}^\gamma u \equiv u$ mod m_L^2

$\Rightarrow \gamma$ acts trivially on $\ell = \mathcal{O}_L/m_L \Rightarrow$ also trivial on $\mu_{|\ell|-1}(L)$. These with u generate the ring $\mathcal{O}_L/m_L^2$.

$\Rightarrow \gamma \in \Gamma_1$.

Conversely $\gamma \in \Gamma_1 \Rightarrow {}^\gamma u \equiv u$ mod $m_L^2 \Rightarrow \frac{{}^\gamma u}{u} \equiv 1 \Rightarrow \gamma \in \ker$ map.

$\Rightarrow$ injective homo $\Gamma_0/\Gamma_1 \hookrightarrow \ell^\times$

(c) $s \geq 1 \Rightarrow \forall \gamma \in \Gamma_s : v_L(\gamma u - u) \geq s+1 \Rightarrow v_L\left(\frac{\gamma u - u}{u^{s+1}}\right) \geq 0 \Rightarrow$ welldefined map $\Gamma_s \to \ell$

$$\gamma, \delta \in \Gamma_s \Rightarrow \underbrace{\frac{\gamma\delta u - u}{u^{s+1}}} = \underbrace{\frac{\gamma(\delta u - u)}{u^{s+1}}} + \underbrace{\frac{\gamma u - u}{u^{s+1}}}$$

$$w := \frac{\delta u - u}{u^{s+1}} \in \mathcal{O}_L$$

$\gamma \in \Gamma_s, s \geq 1 \Rightarrow \gamma w \equiv w \mod m_L$

$\gamma u \equiv u \mod m_L^2$

$$\Rightarrow \frac{\gamma(\delta u - u)}{u^{s+1}} \equiv \frac{\delta u - u}{u^{s+1}} \mod m_L .$$

$\Rightarrow$ Homomorphism.

$\gamma \in \ker \iff v_L(\gamma u - u) \geq s+2$ | γ acts trivially on $\mu_{|\ell|-1}$ and on u mod m_L^{s+2}

$\Longrightarrow$ on $\mathcal{O}_L / m_L^{s+2} \iff \gamma \in \Gamma_{s+1}$.

$\Rightarrow \ker = \Gamma_{s+1} \Rightarrow$ injective $\Gamma_s / \Gamma_{s+1} \hookrightarrow \ell$

qed.

Proposition 11.4.4: (a) The factor group Γ/Γ_0 is cyclic.

(b) The factor group Γ_0/Γ_1 is cyclic of order prime to p.

(c) The subgroup Γ_1 is a p-group.

Because it is a successive extension of finite p-groups.

Corollary 11.4.5: The galois group Γ is solvable.

Proposition 11.4.6: The extension L/K is ...

(a) ... trivial if and only if $\Gamma_{-1} = 1$. $\leftarrow$ because $\Gamma = \Gamma_{-1}$

(b) ... unramified if and only if $\Gamma_0 = 1$. $\leftarrow$ because Γ_0 = inertia group

(c) ... tame if and only if $\Gamma_1 = 1$. $\leftarrow$ because $|\Gamma_0| = e_{L/K}$ is prime to p iff $\Gamma_1 = 1$.

Definition 11.4.7: (a) The subgroup Γ_1 is called the *wild inertia group* of L/K.

(b) The subfactor group Γ_0/Γ_1 is called the *tame inertia group* of L/K.

Proposition 11.4.8: For any intermediate field K' of L/K and any $s \geqslant -1$, the s-th ramification group of $\Gamma' = \mathrm{Gal}(L/K')$ in its own right is equal to $\Gamma' \cap \Gamma_s$.

$(\Gamma')_s = \Gamma' \cap \Gamma_s$.

$$\begin{tikzcd} L \arrow[d, dash, "\Gamma'"] \\ K' \arrow[d, dash] \\ K \end{tikzcd}$$

Γ

Remark 11.4.9: Thus the lower numbering filtration behaves well with respect to the passage of subgroups. But to extend it to infinite galois groups we must study its behavior under passage to factor groups.

11.5 The upper numbering filtration

We keep the situation of Section 11.4.

$\bar f$ separable $\Rightarrow \bar f'(\beta) \neq 0$

$\Uparrow$

Lemma 11.5.1: There exists an element $b \in L$ such that $\mathcal{O}_L = \mathcal{O}_K[b]$.

Proof: l/k is generated by some $\beta \in l$. Let $\bar f \in k[X]$ be its min. pol., say $d = \deg(\bar f)$.

Let $f \in \mathcal{O}_K[X]$ be any lift of $\bar f$.

Choose $b \in \mathcal{O}_L$ with $[b] = \beta$.

Look at $f(b) \in \mathfrak{m}_L$. $\boxed{\text{If } v_L(f(b)) = 1}$ then $\mathcal{O}_L = \mathcal{O}_K[b]$.

If not, take $b + \pi_L$ instead:

$\Rightarrow f(b+\pi_L) = f(b) + f'(b)\cdot \pi_L + O(\pi_L^2)$

$\equiv f'(b)\cdot \pi_L \mod \mathfrak{m}_L^2$

$(f'(b) \mod \mathfrak{m}_L) = \bar f'(\beta) \neq 0$ in l

$\Rightarrow$ indeed $v(f(b+\pi_L)) = 1$

ok.

$\Rightarrow \{b^i \mid 0 \le i < d\}$ generates l over k

$\Rightarrow \{b^i \mid 0 \le i < d\} \cdot \{f(b)^j \mid 0 \le j < e_{L/K}\}$ form a basis of $\mathcal{O}_L$ over $\mathcal{O}_K$.

$\mathcal{O}_L/\mathfrak{m}_K \cdot \mathcal{O}_L = \mathcal{O}_L/\mathfrak{m}_L^{e_{L/K}}$ successive extension of $\mathfrak{m}_L^j/\mathfrak{m}_L^{j+1}$ for $0 \le j < e_{L/K}$

and $\dim_k(\mathcal{O}_L/\mathfrak{m}_K\mathcal{O}_L) = [K/L]$.

qed.

Fix b as above.

Definition 11.5.2: For any $\gamma \in \Gamma$ we set $i_{L/K}(\gamma) := v_L({}^\gamma b - b)$.

Lemma 11.5.3: For any $\gamma \in \Gamma$ and any s we have $\gamma \in \Gamma_s$ if and only if $i_{L/K}(\gamma) \geqslant s+1$.

In particular $i_{L/K}(\gamma)$ is independent of the choice of b.

direct consequ. of def.

i.e.: $\forall c \in \mathcal{O}_L$: ${}^\gamma c - c \equiv 0$ and ${}^\gamma b - b$.

${}^\gamma c \equiv c$

Now consider an intermediate field L' of L/K which is galois over K. Let π denote the canonical projection $\Gamma \overset{\pi}{\twoheadrightarrow} \Gamma' := \mathrm{Gal}(L'/K)$ with kernel $\Delta := \mathrm{Gal}(L/L')$.

$$\left.\begin{array}{l} L \\ \Delta \,\big| \\ L' \\ \Gamma' \,\big| \\ K \end{array}\right\} \Gamma$$

Proposition 11.5.4: For any $\gamma' \in \Gamma'$ we have

$$\boxed{i_{L'/K}(\gamma') = \frac{1}{e_{L/L'}} \cdot \sum_{\gamma \in \pi^{-1}(\gamma')} i_{L/K}(\gamma).}$$

Proof: Write $\mathcal{O}_L = \mathcal{O}_K[b]$ as above $\Rightarrow i_{L/K}(\gamma) = v_L({}^{\gamma}b - b)$.

and $\mathcal{O}_{L'} = \mathcal{O}_K[b']$ $\Rightarrow i_{L'/K}(\gamma') = v_{L'}({}^{\gamma'}b' - b') = \frac{1}{e_{L/L'}} v_L({}^{\gamma'}b' - b')$.

To be proven: $\frac{1}{e_{L/L'}} \cdot v_L({}^{\gamma'}b' - b') = \frac{1}{e_{L/L'}} \cdot \sum_{\gamma \mapsto \gamma'} v_L({}^{\gamma}b - b)$

$\Leftrightarrow$ ${}^{\gamma'}b' - b'$ differs from $\prod_{\gamma \mapsto \gamma'} ({}^{\gamma}b - b)$ by a unit in $\mathcal{O}_L^\times$.

Fix any $\gamma \mapsto \gamma'$,

$\Leftrightarrow$ ${}^{\gamma'}b' - b'$ differs from $\prod_{\delta \in \Delta} ({}^{\gamma\delta}b - b)$ by a unit in $\mathcal{O}_L^\times$.

$\Leftrightarrow$ each of them divides the other in $\mathcal{O}_L$.

Let $f \in \mathcal{O}_{L'}[X]$ be the min. pol. of b over L'. $\xRightarrow{L = L'(b)} f(X) = \prod_{\delta \in \Delta} (X - {}^{\delta}b)$

Let ${}^{\gamma}f$ be obtained from f by applying γ to its coefficients. $\Rightarrow ({}^{\gamma}f)(X) = \prod_{\delta \in \Delta} (X - {}^{\gamma\delta}b)$

Plug in $X = b$

11.5.3 $\Rightarrow \forall c \in \mathcal{O}_{L'}$: ${}^{\gamma'}c \equiv c \mod ({}^{\gamma'}b' - b')$

$\Rightarrow$ $\sigma f = \sigma' f \equiv f$ and $(\sigma' b' - b')$

$\Rightarrow$ $\underline{\prod_{\delta \in \Delta} (\sigma^{\delta} b - b)} = \pm (\sigma f)(b) \equiv \pm f(b) = 0$ and $\underline{(\sigma' b' - b')}$

Write $\underline{b' = g(b)}$ with $g \in \underline{O_K[X]}$.

$\Rightarrow$ b is a zero of the polynomial $g(X) - b' \in O_{L'}[X]$.

$\Rightarrow$ $g(X) - b' = f(X) \cdot h(X)$ for some $\underline{h \in O_{L'}[X]}$.

Apply σ $\Rightarrow$ $g(X) - \sigma' b' = (\sigma f)(X) \cdot (\sigma h)(X)$

Plug in b $\Rightarrow$ $\underbrace{g(b) - \sigma' b'}_{b' - \sigma' b'} = (\sigma f)(b) \cdot \underbrace{(\sigma h)(b)}_{\in O_L} \equiv 0$ and $(\sigma f)(b) = \pm \prod_{\delta} (\sigma^{\delta} b - b)$

qed.