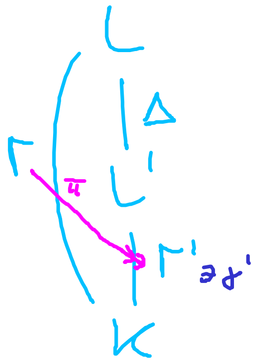
Reminder:

We fix a nonarchimedean local field K with normalized valuation v and valuation ring $\mathcal{O}$ and finite residue field $k = \mathcal{O}/\mathfrak{m}$ of characteristic p. Take a finite galois extension L/K with galois group Γ and residue field $\ell = \mathcal{O}_L/\mathfrak{m}_L$. Let v_L denote the normalized valuation on L. Consider an intermediate field L' of L/K which is galois over K. Let π denote the canonical projection $\Gamma \twoheadrightarrow \Gamma' := \mathrm{Gal}(L'/K)$ with kernel $\Delta := \mathrm{Gal}(L/L')$.

Definition 11.4.1: For every real number $s \geqslant -1$ the *s-th ramification group of L/K in the lower numbering* is

$$\Gamma_s \ := \ \big\{\gamma \in \Gamma \;\big|\; \forall\, a \in \mathcal{O}_L \colon v_L({}^\gamma a - a) \geqslant s+1\big\}.$$



Lemma 11.5.1: There exists an element $b \in L$ such that $\mathcal{O}_L = \mathcal{O}[b]$.

Definition 11.5.2: For any $\gamma \in \Gamma$ we set $i_{L/K}(\gamma) := v_L({}^\gamma b - b)$.

Lemma 11.5.3: For any $\gamma \in \Gamma$ and any s we have $\gamma \in \Gamma_s$ if and only if $i_{L/K}(\gamma) \geqslant s+1$.

Proposition 11.5.4: For any $\gamma' \in \Gamma'$ we have

$$\boxed{i_{L'/K}(\gamma') \ = \ \frac{1}{e_{L/L'}} \cdot \sum_{\gamma \in \pi^{-1}(\gamma')} i_{L/K}(\gamma).}$$

$$[\Gamma_0 : \Gamma_x] = \frac{|\Gamma_0|}{|\Gamma_x|}$$

Construction 11.5.5: We are interested in the function

$$\boxed{\eta_{L/K} \colon [-1,\infty[\longrightarrow [-1,\infty[, \quad s \mapsto \int_0^s \frac{dx}{[\Gamma_0 : \Gamma_x]}.}$$

$-1 < x < 0$: $\Gamma_x = \Gamma_0$

Here for $s < 0$ we interpret $\int_0^s$ as $-\int_s^0$, and $[\Gamma_0 : \Gamma_x]$ as $[\Gamma_x : \Gamma_0]^{-1}$ for $x < 0$.

Proposition 11.5.6: The function $\eta_{L/K}$ is strictly monotone increasing and bijective.

continuous.

$$\eta_{L/K}(-1) = \int_0^{-1} \frac{dx}{[\Gamma_0 : \Gamma_x]} = -\int_{-1}^{0} \frac{dx}{1} = -1 \quad \text{and} \quad \eta_{L/K}' \geq \frac{1}{|\Gamma|} \text{ everywhere}$$

$$\Rightarrow \lim_{s \to \infty} \eta_{L/K}(s) = \infty.$$

Proposition 11.5.7: For any $s \in [-1,\infty[$ we have

$$\boxed{\eta_{L/K}(s) = \frac{1}{|\Gamma_0|} \cdot \sum_{\gamma \in \Gamma} \min\{i_{L/K}(\gamma), s+1\} - 1.} =: \Theta(s) \text{ continuous}$$

Proof: $\Theta(0) = \frac{1}{|\Gamma_0|} \cdot \sum_{\gamma\in\Gamma} \min\{i_{L/K}(\gamma), 1\} - 1 = \frac{1}{|\Gamma_0|} \cdot \#\underbrace{\{\gamma \in \Gamma \mid i_{L/K}(\gamma) \geq 1\}}_{\Gamma_0} - 1$

$= \frac{|\Gamma_0|}{|\Gamma_0|} - 1 = 0 = \eta_{L/K}(0)$

$\forall s \notin \mathbb{Z}$:

$$\Theta'(s) = \frac{1}{|\Gamma_0|} \cdot \sum_{\gamma\in\Gamma} \begin{cases} 1 & \text{if } i_{L/K}(\gamma) \geq s+1 \\ 0 & \text{else} \end{cases}$$

$$= \frac{|\Gamma_s|}{|\Gamma_0|} = \frac{1}{[\Gamma_0 : \Gamma_s]} = \eta_{L/K}'(s)$$

$\Rightarrow \eta_{L/K} = \Theta$. qed.

Theorem 11.5.8: *(Herbrand)* For any $s \in [-1, \infty[$ we have $\boxed{\pi(\Gamma_s) = \Gamma'_{\eta_{L/L'}(s)}}$.

$$\begin{array}{c} L \\ \Delta \Big| \\ L' \\ \Big| \\ K \end{array}$$

Proof: Take $\gamma' \in \Gamma'$ and choose $\gamma \in \Gamma$ with $\gamma \mapsto \gamma'$ and $i_{L/K}(\gamma)$ is maximal.

Claim: $i_{L'/K}(\gamma') - 1 = \eta_{L/L'}(i_{L/K}(\gamma) - 1)$.

Proof: Set $m := i_{L/K}(\gamma)$. Then $\forall \delta \in \Delta$:

either $i_{L/K}(\delta) \geq m \Rightarrow i_{L/K}(\gamma\delta) \geq m \Rightarrow i_{L/K}(\gamma\delta) = m$ by the choice of γ.

or $i_{L/K}(\delta) < m \Rightarrow i_{L/K}(\gamma\delta) = i_{L/K}(\delta) < m$

$\Rightarrow$ In both cases $i_{L/K}(\gamma\delta) = \min\{i_{L/K}(\delta), m\}$.

$$\Rightarrow i_{L'/K}(\gamma') \overset{11.5.4}{=} \frac{1}{e_{L/L'}} \cdot \sum_{\delta \in \Delta} i_{L/K}(\gamma\delta) = \frac{1}{e_{L/L'}} \cdot \sum_{\delta \in \Delta} \min\{i_{L/K}(\delta), m\}$$

$$= \frac{1}{|\Delta_0|} \cdot \sum_{\delta \in \Delta} \min\{i_{L/L'}(\delta), \underset{s+1}{\underset{\|}{m}}\}$$

$$\overset{11.5.7 \text{ for } L/L'}{=} \eta_{L/L'}(m-1) + 1 \qquad \text{qed.}$$

So $\forall s \geq -1$:

$$\gamma' \in \pi(\Gamma_s) \underset{\text{max. of } \gamma}{\Longleftrightarrow} \gamma \in \Gamma_s \overset{11.5.3}{\Longleftrightarrow} i_{L/K}(\gamma) - 1 \geq s \underset{\text{strict monotone}}{\Longleftrightarrow} \eta_{L/L'}(i_{L/K}(\gamma) - 1) \geq \eta_{L/L'}(s)$$

$$\underset{\text{Claim}}{\Longleftrightarrow} i_{L'/K}(\gamma') - 1 \geq \eta_{L/L'}(s) \overset{11.5.3}{\Longleftrightarrow} \gamma' \in \Gamma'_{\eta_{L/L'}(s)}.$$

qed

Proposition 11.5.9: We have $\boxed{\eta_{L/K} = \eta_{L'/K} \circ \eta_{L/L'}.}$

Proof: $\eta_{L/K}(0) = 0 = \eta_{L'/K}(0) = \eta_{L'/K}(\eta_{L/L'}(0))$

For $s \notin \mathbb{Z}$: $\underbrace{\eta'_{L/K}(s) = \frac{1}{[\Gamma_0 : \Gamma_s]} = \frac{|\Gamma_s|}{|\Gamma_0|} = \frac{|\Gamma_s|}{e_{L/K}}}$ (*) With $t := \eta_{L/L'}(s)$ we have

$$1 \longrightarrow \underbrace{\Delta \cap \Gamma_s}_{\Delta_s} \longrightarrow \Gamma_s \longrightarrow \Gamma'_t \longrightarrow 1 \quad \text{exact}$$

$$\Rightarrow \eta'_{L/K}(s) \overset{(*)}{=} \frac{|\Gamma_s|}{e_{L/K}} = \frac{|\Gamma'_t|}{e_{L'/K}} \cdot \frac{|\Delta_s|}{e_{L/L'}} = \eta'_{L'/K}(t) \cdot \eta'_{L/L'}(s)$$

$$= \eta'_{L'/K}(\eta_{L/L'}(s)) \cdot \eta'_{L/L'}(s)$$

$$= (\eta_{L'/K} \circ \eta_{L/L'})'(s). \qquad \Rightarrow \text{qed}.$$

Definition 11.5.10: For any real number $t \geqslant -1$ we define the *t-th ramification group of L/K in the upper numbering* as $\boxed{\Gamma^t := \Gamma_{\eta^{-1}_{L/K}(t)}.}$

Proposition 11.5.11: For any $t \in [-1, \infty[$ we have $\boxed{\pi(\Gamma^t) = (\Gamma')^t.}$

Proof: With $t = \eta_{L/K}(s)$

$\Leftrightarrow \underline{\eta^{-1}_{L/K}(t) = s}$

we get:

$\underline{t = \eta_{L'/K}(s')}$ for $\underline{s' := \eta_{L/L'}(s)}$

and $\pi(\Gamma^t) = \pi(\Gamma_s) = \Gamma'_{s'} = (\Gamma')^t$

qed.

Now consider an arbitrary galois extension L/K which is not necessarily finite.

Definition 11.5.12: For any real number $t \geqslant -1$ we define the *t-th ramification group of L/K* as

$$\mathrm{Gal}(L/K)^t \;:=\; \varprojlim_{L'} \mathrm{Gal}(L'/K)^t,$$

where the limit extends over all intermediate fields L' that are finite and galois over K.

Proposition 11.5.13: For any intermediate field L' of L/K that is galois over K and any real number $t \geqslant -1$ the restriction induces a surjection $\mathrm{Gal}(L/K)^t \twoheadrightarrow \mathrm{Gal}(L'/K)^t$.

Show: $\mathrm{Gal}(L/K) \twoheadrightarrow \mathrm{Gal}(L'/K)$

For all $L''/L'/K$ galois finite:
$\mathrm{Gal}(L''/K)^t \twoheadrightarrow \mathrm{Gal}(L'/K)^t$.

The system is filtered.

Then $\Rightarrow$ $\varprojlim_{L''} \mathrm{Gal}(L''/K)^t \twoheadrightarrow \mathrm{Gal}(L'/K)^t$

Now take $\varprojlim_{L'}$.

qed