Reminder: Let G be a finite cyclic group of order n with a generator g. For a commutative ring R we are interested in the ideal $I_G := (g-1)$ of the group ring $R[G]$ and the element

$=(1-g)$

$$N_G \; := \; \sum_{g' \in G} g' \; \in \; R[G].$$

Definition 12.1.1: For any $R[G]$-module M we define the *Tate cohomology* groups:

$$\begin{aligned} \hat{H}^0(G,M) \; &:= \; M^G/N_G M, \\ \hat{H}^{-1}(G,M) \; &:= \; \ker(N_G|M)/I_G M. \end{aligned}$$

Definition 12.1.2: If these are both finite, the *Herbrand quotient* of M is defined as

$$h(G,M) \; := \; \frac{|\hat{H}^0(G,M)|}{|\hat{H}^{-1}(G,M)|}.$$

Consider a finite cyclic field extension L/K with Galois group Γ. For any representation M of Γ we write

$$\hat{H}^i(L/K,M) \; := \; \hat{H}^i(\Gamma,M).$$

Proposition 12.2.2: We have $\hat{H}^i(L/K,L) = 0$ for each i.

Proposition 12.2.3: We have $\hat{H}^{-1}(L/K, L^\times) = 1$. *(Hilbert theorem 90)*

Proof: Let σ be a generator of $\Gamma = Gal(L/K)$, $|\Gamma| = n$

Take $x \in L^\times$ with $N_\Gamma(x) = \prod_{i=0}^{n-1} \sigma^i x = 1$.

To show: $\exists y \in L^\times : \frac{y}{\sigma y} = x$. $\Leftrightarrow$ $\boxed{y = x \cdot \sigma y}$

Take $\sum_{i=0}^{n-1} \left(\prod_{0 \le j < i} \sigma^j x \right) \cdot \sigma^i \in End_{K\text{-Vectorspace}}(L)$

This is $\neq 0$, because the σ^i are K-linearly independent.

So $\exists z \in L \setminus \{0\}$: $y := \sum_{i=0}^{n-1} \left(\prod_{0 \le j < i} \sigma^j x \right) \cdot \sigma^i(z) \in L^\times$.

$\boxed{x \cdot \sigma y} = x \cdot \sum_{i=0}^{n-1} \left(\prod_{0 \le j < i} \sigma^{j+1} x \right) \cdot \sigma^{i+1}(z) = \sum_{i=0}^{n-1} \left(\prod_{0 \le j \le i} \sigma^j x \right) \cdot \sigma^{i+1}_z$

$= \sum_{i=1}^{n} \left(\prod_{0 \le j < i} \sigma^j x \right) \cdot \sigma^i_z = \boxed{y}$

$\prod_{0 \le j < n} \sigma^j x \cdot \sigma^n_z = N_\Gamma(x) \cdot z = z$

$y = \sum a_i \sigma^i_z$

qed.

Proposition 12.2.4: We have $\hat{H}^i(L/K, L^\times) = 1$ for each i if K is finite.

$\Downarrow$ $L^\times$ finite.

Proof: $\hat{H}^{-1} = 1$ by 12.2.3

$|\hat{H}^0| = |\hat{H}^{-1}|$ by 12.1.4

qed.

For the rest of this chapter we assume that K is a nonarchimedean local field with normalized valuation v_K and valuation ring $\mathcal{O}_K$ and residue field $k = \mathcal{O}_K/\mathfrak{m}_K$ of characteristic p. We let v_L denote the normalized valuation on L and $\ell = \mathcal{O}_L/\mathfrak{m}_L$ its residue field.

Proposition 12.2.5: If L/K is unramified, then:

(a) The norm map $\mathrm{Nm}_{L/K} \colon \mathcal{O}_L^\times \to \mathcal{O}_K^\times$ is surjective.

(b) We have $\hat{H}^i(L/K, \mathcal{O}_L^\times) = 1$ for each i.

(c) We have $\hat{H}^0(L/K, L^\times) \cong \mathbb{Z}/[L/K]\mathbb{Z}$.

$$\hat H^0 = (\mathcal{O}_L^\times)^\Gamma / N_\Gamma(\mathcal{O}_L^\times) = \mathcal{O}_K^\times / \mathrm{Nm}_{L/K}(\mathcal{O}_L^\times)$$

So (a) $\Longleftrightarrow$ $\hat H^0 = 1$. $\Longleftarrow$ (b)

trivial Γ-action!

$$1 \to \mathcal{O}_L^\times \to L^\times \xrightarrow{v_L} \mathbb{Z} \to 0$$

exact $\Rightarrow$ exact

$$\begin{array}{ccccc} \overset{(b)}{0 =} \boxed{\hat H^0(\mathcal{O}_L^\times)} & \to & \hat H^0(L^\times) & \xrightarrow{\sim} & \boxed{\hat H^0(\mathbb{Z})} \\ \uparrow & & & & \downarrow \\ \boxed{\hat H^{-1}(\mathbb{Z})} & \xleftarrow{\sim} & \boxed{\hat H^{-1}(L^\times)} & \leftarrow & \hat H^{-1}(\mathcal{O}_L^\times) \overset{(b)}{=} 0 \end{array}$$

order n $(*)$

$\overset{12.2.4}{=} 0$

$\Rightarrow \hat H^{-1}(\mathbb{Z}) = 0$ and

$\hat H^0(L^\times) \cong \hat H^0(\mathbb{Z}) = \mathbb{Z}^\Gamma / N_\Gamma \mathbb{Z} = \mathbb{Z}/n\mathbb{Z}$

$n = [L/K]$.

So (b) $\Rightarrow$ (c).

Proof of (b): For any $n \geq 0$ set

$$U_L^n := \{x \in \mathcal{O}_L^\times \mid x \equiv 1 \bmod \mathfrak{m}_L^n\}$$

$\Rightarrow U_L^0/U_L^1 \cong \ell^\times$ with $\Gamma \cong \mathrm{Gal}(\ell/k)$

$\overset{12.2.4}{\Longrightarrow} \hat H^i(U_L^0/U_L^1) = 1$ for all i.

$\forall n \geq 1$: $U_L^n/U_L^{n+1} \xleftarrow{\sim} \ell = \mathcal{O}_L/\mathfrak{m}_L$

$[1 + \pi^n \cdot y] \longleftarrow y$, $y \in \ell$

for $\pi \in \mathfrak{m}_K$ uniformizer.

$\Rightarrow \pi$ uniformizer of L.

$\Rightarrow \hat H^i(U_L^n/U_L^{n+1}) \cong \hat H^i(\ell) = 0$ for all i.

Induction $\Rightarrow \forall n \geq 0: \forall i: \hat H^i(\mathcal{O}_L^\times/U_L^n) = 1$

$$\mathcal{O}_L^\times = \varprojlim_n (\mathcal{O}_L^\times/U_L^n)$$

Take $x \in \mathcal{O}_K^\times = (\mathcal{O}_L^\times)^\Gamma$

Then $\forall n \geq 0$: $[x] = 1$ in $\hat{H}^0(\mathcal{O}_L^\times / U_L^n)$

i.e. $\exists\, y_n \in \mathcal{O}_L^\times$: $x \equiv N_{L/K}\, y_n$ mod $\mathfrak{m}_L^n$.

..... $\Rightarrow \exists\, y \in \mathcal{O}_L^\times$: $x = N_{L/K}\, y$ i.e. $[x] = 1$ in $\hat{H}^0(\mathcal{O}_L^\times)$

So $\hat{H}^0(\mathcal{O}_L^\times) = 1$.

By $(*)$ we have $1 \to \hat{H}^0(L^\times) \xrightarrow[\sim]{} \mathbb{Z}/n\mathbb{Z} \to \hat{H}^{-1}(\mathcal{O}_L^\times) \to 1$

Also $L^\times = \pi^{\mathbb{Z}} \times \mathcal{O}_L^\times \cong \mathbb{Z} \times \mathcal{O}_L^\times$ $\Rightarrow \hat{H}^{-1}(\mathcal{O}_L^\times) = 1$.

trivial action

$\Rightarrow \hat{H}^0(L^\times) \cong \hat{H}^0(\mathbb{Z}) \times \hat{H}^0(\mathcal{O}_L^\times)$

$\cong \mathbb{Z}/n\mathbb{Z}$

qed.

Proposition 12.2.6: For any cyclic L/K we have

$$\left|\hat{H}^0(L/K, L^\times)\right| = [L/K].$$

Proof: Claim: $h(\mathcal{O}_L^\times) = 1$.

if this holds:

$$1 \to \mathcal{O}_L^\times \to L^\times \xrightarrow{v_L} \mathbb{Z} \to 0$$

$$\Rightarrow h(L^\times) = h(\mathcal{O}_L^\times) \cdot \underbrace{h(\mathbb{Z})}_{=[L/K]} = [L/K]$$

$$\hat{H}^{-1}(L^\times) = 1$$

$$\Rightarrow \left|\hat{H}^0(L^\times)\right| = [L/K].$$

By 12.1.4 and 12.1.6 it suffices to show:

$\exists$ Γ-invariant open subgroup $V < \mathcal{O}_L^\times$ with $h(V) = 1$.

If $\mathrm{char}(K) = 0$, then $\exists\, n$:

$$U_L^n \underset{\exp}{\overset{\log}{\rightleftarrows}} (\mathfrak{m}_L^n, +) \quad (\sim)$$

Let $\{\sigma b \mid \sigma \in \Gamma\}$ be a normal basis of L over K.

Replace b by ab for $a \in \mathcal{O}_K$ with $v(a) \gg 0$

$\Rightarrow b \in \mathfrak{m}_L^n$ and $W := \bigoplus_{i=0}^{n-1} \mathcal{O}_K \cdot \sigma^i b$ is an open subgroup of $\mathfrak{m}_L^n$.

Then W is free over $\mathcal{O}_K[\Gamma]$. $\Rightarrow h(W) = 1$.

$\Rightarrow \exp(W) =: V$ does the job.

qed.

12.3 The reciprocity isomorphism

Now we fix a maximal unramified extension $\tilde{K}/K$. We take a finite Galois extension L/K and set $\tilde{L} := L\tilde{K}$. Then $\tilde{L}/K$ is Galois and we have a natural surjection

$$d_K\colon \operatorname{Gal}(\tilde{L}/K) \twoheadrightarrow \operatorname{Gal}(\tilde{K}/K) \cong \operatorname{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}.$$

We are interested in the set of *generalized Frobenius elements*

$$\operatorname{Frob}_{\tilde{L}/K} := \{\sigma \in \operatorname{Gal}(\tilde{L}/K) \mid d_K(\sigma) \in \mathbb{Z}^{\geqslant 1}\}.$$

Clearly this subset is closed under multiplication. Take $\sigma \in \operatorname{Frob}_{\tilde{L}/K}$.

Proposition 12.3.1: The subfield $L_\sigma := \tilde{L}^{\langle\sigma\rangle}$ is finite over K and $\tilde{L}/L_\sigma$ is unramified. with Frobenius element σ.

Proof: $\operatorname{Gal}(\tilde{L}/K) < \underbrace{\operatorname{Gal}(L/K)}_{\text{finite}} \times \operatorname{Gal}(\tilde{K}/K) \cong \hat{\mathbb{Z}}$

$\operatorname{Gal}(\tilde{L}/L_\sigma) = \overline{\langle\sigma\rangle} \xrightarrow{\sim} d_K(\sigma)\cdot\hat{\mathbb{Z}}$, $\quad d_K(\sigma) \in \hat{\mathbb{Z}}$

qed.

Proposition 12.3.2: The residue class

$$r_{\tilde{L}/K}(\sigma) := [\operatorname{Nm}_{L_\sigma/K}(\pi_{L_\sigma})] \in K^\times/\operatorname{Nm}_{L/K}\mathcal{O}_L^\times$$

is independent of the choice of a uniformizer π_{L_σ} of L_σ.

Pf: Equivalent: $\operatorname{Nm}_{L_\sigma/K}(\mathcal{O}_{L_\sigma}^\times) \subset \operatorname{Nm}_{L/K}(\mathcal{O}_L^\times)$.

$\operatorname{Nm}_{L_\sigma/K}(\mathcal{O}_{L_\sigma}^\times) = \operatorname{Nm}_{LL_\sigma/K}(\mathcal{O}_{LL_\sigma}^\times) = \operatorname{Nm}_{L/K}\operatorname{Nm}_{LL_\sigma/L}\mathcal{O}_{LL_\sigma}^\times \subset \operatorname{Nm}_{L/K}(\mathcal{O}_L^\times)$

$LL_\sigma \subset \tilde{L}$ unramified over L_σ $\Longrightarrow \operatorname{Nm}_{LL_\sigma/L_\sigma}\mathcal{O}_{LL_\sigma}^\times = \mathcal{O}_{L_\sigma}^\times$.

qed.

Proposition 12.3.3: This defines a multiplicative map

$$r_{\tilde{L}/K}\colon \operatorname{Frob}_{\tilde{L}/K} \longrightarrow K^\times/\operatorname{Nm}_{L/K}\mathcal{O}_L^\times.$$

See Neukirch Ch. IV §5.

$$\begin{tikzcd} \tilde{L} \arrow[d, dash] \arrow[dr, dash] \arrow[ddr, dash, "\tilde{\sigma}"] & \\ L \arrow[dr, dash, "\sigma"'] & \tilde{K} \arrow[d, dash] \\ & K \end{tikzcd}$$

Theorem 12.3.4: There exists a unique homomorphism (12.3.3)

$$r_{L/K}\colon \operatorname{Gal}(L/K) \longrightarrow K^\times/\operatorname{Nm}_{L/K} L^\times$$

$\sigma \mapsto$

with

$$r_{L/K}(\sigma) = [\operatorname{Nm}_{L_{\tilde{\sigma}}/K}(\pi_{L_{\tilde{\sigma}}})]$$

for any $\tilde{\sigma} \in \operatorname{Frob}_{\tilde{L}/K}$ with $\tilde{\sigma}|L = \sigma$ and any uniformizer $\pi_{L_{\tilde{\sigma}}}$ of $L_{\tilde{\sigma}}$.

Proof: Well-definedness: Let $\tilde{\sigma}, \tilde{\sigma}'$ be two lifts of σ

$\Rightarrow$ they differ by an elem't of $\operatorname{Gal}(\tilde{L}/L)$ = unramified

So if $d_K(\tilde{\sigma}) = d_K(\tilde{\sigma}')$ then $\tilde{\sigma} = \tilde{\sigma}'$ $\to$ OK.

Else WLOG: $d_K(\tilde{\sigma}) < d_K(\tilde{\sigma}')$. Then $\tilde{\sigma}' = \tilde{\sigma}\cdot\tilde{\tau}$ for $\tilde{\tau} \in \operatorname{Frob}_{\tilde{L}/K}$ with $\tilde{\tau}|L = \mathrm{id}$.

So $L \subset L_{\tilde{\tau}}$ $\Rightarrow$ $r_{\tilde{L}/K}(\tilde{\tau}) = [\operatorname{Nm}_{L_{\tilde{\tau}}/K}\pi_{L_{\tilde{\tau}}}] = [\operatorname{Nm}_{L/K}\operatorname{Nm}_{L_{\tilde{\tau}}/L}\pi_{L_{\tilde{\tau}}}] = [1]$ in $K^\times/\operatorname{Nm}_{L/K}L^\times$.

$\Rightarrow$ $r_{\tilde{L}/K}(\tilde{\sigma}') = r_{\tilde{L}/K}(\tilde{\sigma}\cdot\tilde{\tau}) \overset{12.3.3}{=} r_{\tilde{L}/K}(\tilde{\sigma})\cdot r_{\tilde{L}/K}(\tilde{\tau}) \equiv r_{\tilde{L}/K}(\tilde{\sigma})$ mod $\operatorname{Nm}_{L/K}L^\times$. qed

Proposition 12.3.5: The map $r_{L/K}$ is an isomorphism if L/K is unramified.

Then $N_{L/K}(\mathcal{O}_L^\times) = \mathcal{O}_K^\times$

$\Rightarrow N_{L/K}(L^\times) = \mathcal{O}_K^\times \cdot \pi^{n\mathbb{Z}}$ $\quad n = [L/K]$, $\pi \in K$ unifor.

$$\begin{array}{ccccc} \mathrm{Gal}(L/K) & \xrightarrow{\sim} & K^\times / N_{L/K} L^\times & \xrightarrow[v]{\sim} & \mathbb{Z}/n\mathbb{Z} \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ \mathrm{Frob}_K & \longmapsto & [\pi] & \longmapsto & [1] \end{array}$$

Proposition 12.3.6: Consider finite extensions $L'/L/K$ and $L'/K'/K$ such that both L/K and L'/K' are Galois. Then we have the following commutative diagram:

$$
\begin{tikzcd}
\mathrm{Gal}(L'/K') \arrow[r, "r_{L'/K'}"] \arrow[d, "\mathrm{res}"] & K'^{\times}/\,\mathrm{Nm}_{L'/K'} L'^{\times} \arrow[d, "{[\mathrm{Nm}_{K'/K}]}"] \\
\mathrm{Gal}(L/K) \arrow[r, "r_{L/K}"] & K^{\times}/\,\mathrm{Nm}_{L/K} L^{\times}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & L' \arrow[dl, dash, "\text{galois}"'] \arrow[dr, dash] & \\
K' \arrow[dr, dash] & & L \arrow[dl, dash, "\text{galois}"] \\
 & K &
\end{tikzcd}
$$

$$
\begin{aligned}
&\mathrm{Nm}_{K'/K}\mathrm{Nm}_{L'/K'} L'^{\times} \\
&\quad \| \\
&\mathrm{Nm}_{L'/K} L'^{\times} \\
&\quad \| \\
&\mathrm{Nm}_{L/K}\mathrm{Nm}_{L'/L} L'^{\times} \\
&\quad \cap \\
&\mathrm{Nm}_{L/K} L^{\times}
\end{aligned}
$$

Theorem 12.3.7: The map $r_{L/K}$ induces an isomorphism

$$\underline{r_{L/K}\colon\ \mathrm{Gal}(L/K)_{\mathrm{ab}} \xrightarrow{\sim} K^\times/\,\mathrm{Nm}_{L/K} L^\times} \quad = \hat{H}^0(L/K, L^\times)$$

for the maximal abelian quotient $\mathrm{Gal}(L/K)_{\mathrm{ab}}$ of $\mathrm{Gal}(L/K)$.