Reminder:

Let K be a nonarchimedean local field with normalized valuation v_K and valuation ring $\mathcal{O}_K$ and residue field $k = \mathcal{O}_K/\mathfrak{m}_K$ of characteristic p. Let L/K be a finite Galois extension with Galois group Γ.

Theorems 12.3.4 and 12.3.7: There exists a natural isomorphism

$$\boxed{r_{L/K}\colon\ \mathrm{Gal}(L/K)_{\mathrm{ab}} \xrightarrow{\sim} K^\times/\,\mathrm{Nm}_{L/K}\, L^\times}$$

a^G

G_a

12.4 The existence theorem

Proposition 12.4.1: For any finite Galois extension L/K the subgroup $\mathrm{Nm}_{L/K}\, L^\times$ is closed of finite index in $K^\times$.

$\Rightarrow$ open.

Proof: $\mathcal{O}_K^\times < K^\times$ open

$(\mathrm{Nm}_{L/K} L^\times) \cap \mathcal{O}_K^\times = \mathrm{Nm}_{L/K} \mathcal{O}_L^\times$

$\underbrace{\mathcal{O}_L^\times}_{\text{compact}} \xrightarrow{\mathrm{Nm}_{L/K}} \underbrace{\mathcal{O}_K^\times}_{\text{Hausdorff}} \Rightarrow$ closed.

$L^\times = \pi_L^{\mathbb{Z}} \times \mathcal{O}_L^\times$

$\Rightarrow \mathrm{Nm}_{L/K} L^\times = \underbrace{(\mathrm{Nm}_{L/K} \pi_L)^{\mathbb{Z}} \times \mathrm{Nm}_{L/K} \mathcal{O}_L^\times}_{\text{direkt}}$

$\Rightarrow \mathrm{Nm}_{L/K}$ closed.

qed.

Theorem 12.4.2: For any closed subgroup $H \subset K^\times$ of finite index there exists a finite Galois extension L/K with $\mathrm{Nm}_{L/K} L^\times \subset H$.

(Proof only for $\mathrm{char}(K) = 0$.)

Proof: Let $n := [K^\times : H]$. Then $(K^\times)^n < H$.

Set $K_1 := K(\mu_n)$. Then $K_1^\times \cong \mathbb{Z} \times (\text{finite}) \times \mathbb{Z}_p^{[K_1/\mathbb{Q}_p]}$

$\Rightarrow K_1^\times/(K_1^\times)^n$ is finite

Let $L := K_1(\sqrt[n]{K_1^\times})$. galois over K.

Kummer Theory 10.4.3: $\mathrm{Gal}(L/K_1) \cong \mathrm{Hom}(K_1^\times, \mu_n) = \mathrm{Hom}(K_1^\times/(K_1^\times)^n, \mu_n)$

$\Rightarrow |\mathrm{Gal}(L/K_1)| = |K_1^\times/(K_1^\times)^n|$ finite annihilated by n.

12.3.7 for L/K_1: $\mathrm{Gal}(L/K_1) \xrightarrow{\sim} \underbrace{K_1^\times/\mathrm{Nm}_{L/K_1} L^\times}$

so this is annihilated by n

$\Rightarrow$ quotient of $K_1^\times/(K_1^\times)^n$

is iso!

$\Rightarrow \mathrm{Nm}_{L/K_1} L^\times = (K_1^\times)^n$.

$\Rightarrow \mathrm{Nm}_{L/K} L^\times = \mathrm{Nm}_{K_1/K}((K_1^\times)^n) < (K^\times)^n < H$.

qed.

Conclusion: With H and L as above we get

$$\begin{tikzcd}
\mathrm{Gal}(L/K)^{ab} \arrow[r, "\sim", "r_{L/K}"'] \arrow[d, two heads] & K^\times / \mathrm{Nm}_{L/K} L^\times \arrow[d, two heads] \\
\mathrm{Gal}(K_H/K) \arrow[r, "\sim", "r_{K_H/K}"'] & K^\times / H
\end{tikzcd}$$

$\exists!$ abelian sub extension $K_H \subset L$

L_1, L_2 abelian finite $/K$

$L = L_1 L_2$

$$\begin{tikzcd}
\mathrm{Gal}(L_1L_2/K) \arrow[r, "\sim"] \arrow[d] & K^\times / \mathrm{Nm}_{L/K} L^\times \arrow[d] \\
\mathrm{Gal}(L_i/K) \arrow[r] & \underbrace{K^\times / \mathrm{Nm}_{L_i/K} L_i^\times}
\end{tikzcd}$$

L_i unique as subfield of L

depends only on L_i up to isomorphism over K.

Cor: $L_1 = L_2$ within L iff $\mathrm{Nm}_{L_1/K} L_1^\times = \mathrm{Nm}_{L_2/K} L_2^\times$.

And: $L_1 \subset L_2 \iff L = L_2 \iff \mathrm{Nm}_{L/K} L^\times = \mathrm{Nm}_{L_2/K} L_2^\times$.

Theorem 12.4.3: There is a natural isomorphism

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \cong (K^\times)\hat{},$$

where the profinite completion $(K^\times)\hat{}$ is unnaturally isomorphic to $\underbrace{\hat{\mathbb{Z}} \times \mathcal{O}_K^\times}$.

$$\varprojlim_{L/K \text{ finite abelian}} \mathrm{Gal}(L/K) \cong \varprojlim_{\substack{H \subset K^\times \\ \text{open of finite index}}} K^\times/H$$

$K^\times \cong \underbrace{\mathbb{Z}} \times \underbrace{\mathcal{O}_K^\times}$

Corollary 12.4.4: (a) The map $L \mapsto \mathcal{N}_L := \mathrm{Nm}_{L/K} L^\times$ is a bijection from the set of finite abelian extensions of K up to isomorphism to the set of closed subgroups of finite index of $K^\times$.

⟺ open

(b) We have

$$\begin{aligned} L_1 \subset L_2 &\iff \mathcal{N}_{L_1} \supset \mathcal{N}_{L_2}, \\ \mathcal{N}_{L_1 L_2} &= \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}, \quad \text{and} \\ \mathcal{N}_{L_1 \cap L_2} &= \mathcal{N}_{L_1} \mathcal{N}_{L_2}. \end{aligned}$$

$K^\times \cong \mathbb{Z} \times \mathcal{O}_K^\times$

(c) A finite abelian extension L/K is unramified if and only if $\mathcal{O}_K^\times \subset \mathcal{N}_L$.

(d) The isomorphism $r_{L/K}^{-1} \colon K^\times/\mathcal{N}_L \xrightarrow{\sim} \mathrm{Gal}(L/K)$ sends the coset of any uniformizer of K to a Frobenius element.

$$K^\times/\mathcal{N}_{L'} \xrightarrow{\sim} \mathrm{Gal}(L'/K)$$

$$[\pi_K] \longmapsto \mathrm{Frob}$$

L | tot. ram. | L' | unram. | K

13 Global class field theory

places of K

We fix a number field K and let M_K denote the set of absolute values of K up to equivalence. We let S_∞ denote the subset of archimedean absolute values and write $v \in M_K \setminus S_\infty$ for the respective normalized valuation.

13.1 Ideles

Consider a finite abelian extension L/K with Galois group Γ. Let $\Gamma_v < \Gamma$ denote the decomposition group at $v \in M_K$ and $I_v < \Gamma_v$ the inertia group if $v \notin S_\infty$.

Proposition 13.1.1: The embeddings $\Gamma_v \hookrightarrow \Gamma$ and the local reciprocity isomorphisms induce a surjective homomorphism

$$\boxplus_{v\in M_K} K_v^\times / \mathrm{Nm}_{L_v/K_v}(L_v^\times) \xrightarrow{\sim} \boxplus_{v\in M_K} \Gamma_v \overset{\Sigma}{\twoheadrightarrow} \Gamma.$$

$\Gamma_v = \mathrm{Gal}(L_w/K_v)$

$L_v^\times = (L \otimes_K K_v)^\times = \times_{w|v} L_w^\times$

$\mathrm{Nm}_{L_v/K_v} L_v^\times = \mathrm{Nm}_{L_w/K_v} L_w^\times$

Cokernel.

enough to take $\boxplus_{v\in M_K\setminus S} \Gamma_v$ for any finite S.

For almost all $v \notin S_\infty$ $\mathrm{Nm}_{L_v/K_v} L_v^\times \supset O_{K_v}^\times$.

Definition 13.1.2: The group of *ideles of K* (from "*id. el.*" for "*ideal elements*") is the subgroup

$$\boxed{I_K \;:=\; \Big\{(x_v)_v \in \bigtimes_{v\in M_K} K_v^\times \;\Big|\; \forall' v\colon x_v \in \mathcal{O}_{K_v}^\times\Big\}.}$$

It is endowed with the topology for which the subgroups

$$I_K^S \;:=\; \Big\{(x_v)_v \in \bigtimes_{v\in M_K} K_v^\times \;\Big|\; \forall v \notin S\colon x_v \in \mathcal{O}_{K_v}^\times\Big\} \;\cong\; \bigtimes_{v\in S} \overset{U_v}{\overset{\cap}{K_v^\times}} \times \bigtimes_{v\in M_K\smallsetminus S} \overset{U_v}{\overset{\|}{\mathcal{O}_{K_v}^\times}}$$

for all finite subsets $S \subset M_K$ with $S_\infty \subset S$ are open and carry the product topology. Thus for any choice of open subsets $U_v \subset K_v^\times$ such that $U_v = \mathcal{O}_{K_v}^\times$ for almost all v, the subset $U = \bigtimes_v U_v$ is open, and varying the U_v these subsets form a basis for the topology on I_K.

Caution 13.1.3: This topology is *not* induced from the product topology on $\bigtimes_{v\in M_K} K_v^\times$.

Proposition 13.1.4: The local norms induce a continuous homomorphism

$$\mathrm{Nm}_{L/K}\colon I_L \longrightarrow I_K.$$

$$I_L = \bigcup_{\substack{S_\infty \subset S \subset \Pi_K \\ \text{finite}}} I_L^T = \bigcup_S \bigtimes_{w\in T} L_w^\times \times \bigtimes_{w\in \Pi_L\setminus T} \mathcal{O}_{L_w}^\times = \bigcup_S \bigtimes_{v\in S} \Big(\underbrace{\bigtimes_{w|v} L_w^\times}\Big) \times \bigtimes_{v\in \Pi_K\setminus S} \Big(\underbrace{\prod_{w|v} \mathcal{O}_{L_w}^\times}\Big)$$

$T := \{\text{places of } L \text{ above some } v\in S\}$

$$I_K = \bigcup_S I_K^S = \bigcup_S \bigtimes_{v\in S} K_v^\times \times \bigtimes_{v\in \Pi_K\setminus S} \mathcal{O}_{K_v}^\times$$

Proposition 13.1.5: The embeddings $\Gamma_v \hookrightarrow \Gamma$ induce a continuous surjective homomorphism $I_K \twoheadrightarrow \Gamma$ that vanishes on $\mathrm{Nm}_{L/K} I_L$.

S suff. large

$$I_K^S/\mathrm{Nm}_{L/K} I_L^S = \bigtimes_{v\in S}\big(K_v^\times/\mathrm{Nm}_{L_v/K_v} L_v^\times\big) \times \bigtimes_{v\notin S}\big(\underbrace{\mathcal{O}_{K_v}^\times/\mathrm{Nm}_{L_v/K_v}\mathcal{O}_{L_v}^\times}\big)$$

trivial if v unramified.

$$I_K^S \twoheadrightarrow I_K^S/\mathrm{Nm}_{L/K} I_L^S, \qquad \bigtimes_{v\in S} \Gamma_v \xrightarrow{\Sigma} \Gamma$$

Describing all abelian extensions of K thus translates into describing all possible subgroups that can occur as kernel of the surjection $I_K \twoheadrightarrow \Gamma$. Varying L we obtain a surjective homomorphism

$$\boxed{I_K \longrightarrow\!\!\!\!\rightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)}$$

and it is equivalent to describe its kernel.

$K^\times \hookrightarrow I_K \subset \prod_v K_v^\times$

13.2 Idele classes

We identify $K^\times$ with its image in I_K under the diagonal embedding $x \mapsto (x, x, \ldots)$.

Definition 13.2.1: We call $C_K := I_K/K^\times$ the group of *idele classes*, and we endow C_K with the quotient topology induced from I_K.

Proposition 13.2.2: The natural surjective homomorphism

$$\begin{aligned} I_K &\longrightarrow \mathrm{Frac}(\mathcal{O}_K) := \{\text{fractional ideals of } \mathcal{O}_K\}, \\ (x_v)_v &\longmapsto \prod_{v \in M_K \setminus S_\infty} \mathfrak{p}_v^{v(x_v)} \end{aligned}$$

0 for almost all v.

lies in the following natural commutative diagram with exact rows and columns:

Recall:

$1 \to \mathcal{O}_K^\times \to K^\times \to \mathrm{Frac}(\mathcal{O}_K) \to \mathrm{Cl}(\mathcal{O}_K) \to 1$

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d] & K^\times \arrow[r] \arrow[d] & K^\times/\mathcal{O}_K^\times \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I_K^{S_\infty} \arrow[r] \arrow[d] & I_K \arrow[r] \arrow[d] & \mathrm{Frac}(\mathcal{O}_K) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I_K^{S_\infty}/\mathcal{O}_K^\times \arrow[r] \arrow[d] & C_K \arrow[r] \arrow[d] & \mathrm{Cl}(\mathcal{O}_K) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

In particular there is a natural exact sequence

$$1 \longrightarrow \mathcal{O}_K^\times \longrightarrow I_K^{S_\infty} \longrightarrow C_K \longrightarrow \mathrm{Cl}(\mathcal{O}_K) \longrightarrow 1.$$

Definition 13.2.3: The *norm* of an idele $\underline{x} = (x_v)_v \in I_K$ is defined as

$$|\underline{x}| := \prod_{v \notin S_\infty} |k_v|^{-v(x_v)} \cdot \prod_{K_v \cong \mathbb{R}} |x_v| \cdot \prod_{K_v \cong \mathbb{C}} |x_v|^2.$$

$\mathbb{R}/\mathbb{Z} + \sqrt{2}\mathbb{Z}$

The subgroup of ideles of norm 1 is denoted I_K^1.

Theorem 13.2.4: (a) The group $K^\times$ is a discrete subgroup of I_K^1. $\rightsquigarrow$ C_K Hausdorff.

(b) The quotient $C_K^1 := I_K^1 / K^\times$ with its induced topology is compact.

(c) There are topological group isomorphisms $I_K \cong I_K^1 \times \mathbb{R}$ and $C_K \cong C_K^1 \times \mathbb{R}$.