## Reminder:

We fix a number field K and let  $M_K$  denote the set of absolute values of K up to equivalence. Let  $S_{\infty}$  denote the subset of archimedean absolute values and  $v \in M_K \setminus S_{\infty}$  the respective normalized valuation. **Definition 13.1.2:** The group of *ideles of* K (from "*id. el.*" for "*ideal elements*") is the subgroup

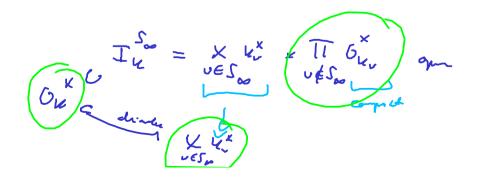
$$I_K := \{ (x_v)_v \in \bigotimes_{v \in M_K} K_v^{\times} \mid \forall' v \colon x_v \in \mathcal{O}_{K_v}^{\times} \}. \qquad \simeq \qquad \prod_{v \in M_K} \{ \mathcal{K}_v^{\times} \mid \mathcal{O}_{\mathcal{K}_v}^{\times} \}.$$

It is endowed with the topology for which the subgroups

$$I_K^S := \left\{ (x_v)_v \in \bigotimes_{v \in M_K} K_v^{\times} \mid \forall v \notin S \colon x_v \in \mathcal{O}_{K_v}^{\times} \right\} \cong \left| \bigotimes_{v \in S} K_v^{\times} \times \bigotimes_{v \in M_K \smallsetminus S} \mathcal{O}_{K_v}^{\times} \right|$$

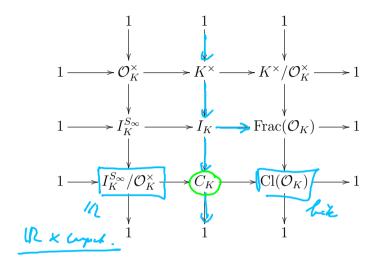
for all finite subsets  $S \subset M_K$  with  $S_{\infty} \subset S$  are open and carry the product topology.

**Definition 13.2.1:** We embed  $\underline{K^{\times}}$  into  $I_K$  via the diagonal embedding  $\underline{x} \mapsto (x, x, \ldots)$ . We call  $C_K := I_K/K^{\times}$  the group of *idele classes*, and endow  $C_K$  with the quotient topology induced from  $I_K$ .



## $\square$

**Proposition 13.2.2:** There is a natural commutative diagram with exact rows and columns:



**Definition 13.2.3:** The norm of an idele  $\underline{x} = (x_v)_v \in I_K$  is defined as

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The subgroup of ideles of norm 1 is denoted  $I_K^1$ .

**Theorem 13.2.4:** (a) The group  $K^{\times}$  is a discrete subgroup of  $I_K^1$ .

(b) The quotient  $C_K^1 := I_K^1/K^{\times}$  with its induced topology is compact.

(c) There are topological group isomorphisms  $I_K \cong I_K^1 \times \mathbb{R}$  and  $C_K \cong C_K^1 \times \mathbb{R}$ .

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## 13.3The reciprocity isomorphism

Now we return to L/K finite abelian with Galois group  $\Gamma$ .

 $C_K / \operatorname{Nm}_{L/K} C_L \xrightarrow{\sim} \Gamma.$ 

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**Theorem 13.3.3:** (a) The map  $L \mapsto \mathcal{N}_L := \operatorname{Nm}_{L/K} C_L$  is a bijection from the set of finite abelian extensions of K up to isomorphism to the set of closed subgroups of finite index of  $C_K$ .

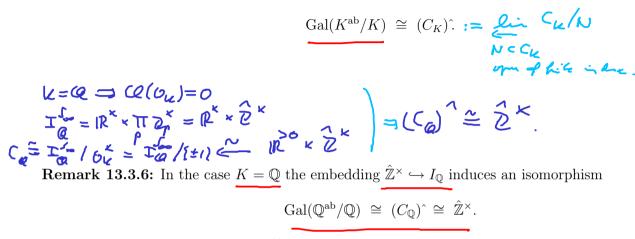
(b) We have

$$\begin{array}{rcl}
L_1 \subset L_2 & \Longleftrightarrow & \mathcal{N}_{L_1} \supset \mathcal{N}_{L_2}, \\
\hline
\mathcal{N}_{L_1 L_2} & = & \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}, \\
\hline
\mathcal{N}_{L_1 \cap L_2} & = & \mathcal{N}_{L_1} \mathcal{N}_{L_2}.
\end{array}$$
and

**Remark 13.3.4:** The local behavior of an abelian extension L/K at a place  $v \in M_K$  can be read off from the global information by pullback under the embedding

$$K_v^{\times} \longleftrightarrow C_K, \quad x_v \mapsto [(1, \dots, 1, x_v, 1, \dots)]$$

Corollary 13.3.5: There is a natural isomorphism



This isomorphism is the reciprocal (!) of that induced by the cyclotomic character:

$$\operatorname{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) = \operatorname{Gal}(\mathbb{Q}(\bigcup_n \mu_n)/\mathbb{Q}) \cong \mathbb{Z}^{\times}.$$