Reminder:

$\coprod$

We fix a number field K and let M_K denote the set of absolute values of K up to equivalence. Let S_∞ denote the subset of archimedean absolute values and $v \in M_K \smallsetminus S_\infty$ the respective normalized valuation.

Definition 13.1.2: The group of *ideles of* K (from "*id. el.*" for "*ideal elements*") is the subgroup

$$I_K \;:=\; \Big\{ (x_v)_v \in \bigtimes_{v\in M_K} K_v^\times \;\Big|\; \forall' v\colon x_v \in \mathcal{O}_{K_v}^\times \Big\}. \qquad = \prod_v{}' \big(K_v^\times \,\big|\, \mathcal{O}_{K_v}^\times \big)$$

It is endowed with the topology for which the subgroups

$$I_K^S \;:=\; \Big\{ (x_v)_v \in \bigtimes_{v\in M_K} K_v^\times \;\Big|\; \forall v \notin S\colon x_v \in \mathcal{O}_{K_v}^\times \Big\} \;\cong\; \boxed{\bigtimes_{v\in S} K_v^\times \times \bigtimes_{v\in M_K\smallsetminus S} \mathcal{O}_{K_v}^\times}$$

for all finite subsets $S \subset M_K$ with $S_\infty \subset S$ are open and carry the product topology.

Definition 13.2.1: We embed $K^\times$ into I_K via the diagonal embedding $x \mapsto (x, x, \ldots)$. We call $C_K := I_K/K^\times$ the group of *idele classes*, and endow C_K with the quotient topology induced from I_K.

$$I_K^{S_\infty} = \underbrace{\bigtimes_{v\in S_\infty} K_v^\times}_{} \times \underbrace{\prod_{v\notin S_\infty} \mathcal{O}_{K_v}^\times}_{\text{compact}} \quad \text{open}$$

$$\mathcal{O}_K^\times \subset \quad \xrightarrow{\text{discrete}} \quad \bigtimes_{v\in S_\infty} K_v^\times$$

Proposition 13.2.2: There is a natural commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d] & K^\times \arrow[r] \arrow[d] & K^\times/\mathcal{O}_K^\times \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I_K^{S_\infty} \arrow[r] \arrow[d] & I_K \arrow[r] \arrow[d] & \mathrm{Frac}(\mathcal{O}_K) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \boxed{I_K^{S_\infty}/\mathcal{O}_K^\times} \arrow[r] \arrow[d] & C_K \arrow[r] \arrow[d] & \boxed{\mathrm{Cl}(\mathcal{O}_K)} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
$$

$\mathbb{R}$

finite

$\mathbb{R} \times$ compact.

Definition 13.2.3: The *norm* of an idele $\underline{x} = (x_v)_v \in I_K$ is defined as

$$|\underline{x}| := \prod_{v \notin S_\infty} |k_v|^{-v(x_v)} \cdot \prod_{K_v \cong \mathbb{R}} |x_v| \cdot \prod_{K_v \cong \mathbb{C}} |x_v|^2.$$

$\rightsquigarrow$ Homo: $I_K \twoheadrightarrow \mathbb{R}^{>0}$

Prop.: $\forall x \in K^\times$: $|(x, \dots, x)| = 1$

The subgroup of ideles of norm 1 is denoted I_K^1.

Theorem 13.2.4: (a) The group $K^\times$ is a discrete subgroup of I_K^1.

(b) The quotient $C_K^1 := I_K^1 / K^\times$ with its induced topology is compact.

(c) There are topological group isomorphisms $I_K \cong I_K^1 \times \mathbb{R}$ and $C_K \cong C_K^1 \times \mathbb{R}$.

Proof: (a) $K^\times \cap I_K^{S_\infty} = \mathcal{O}_K^\times$.

$$\mathop{\times}_{v \in S_\infty} K_v^\times \xrightarrow{(\log|\cdot|^{f_v})} \mathop{\times}_{v \in S_\infty} \mathbb{R} \xrightarrow{\Sigma} \mathbb{R}$$

with compact kernel

$\cup$ subspace of codim 1

$H = \ker(\Sigma)$

$f_v = \begin{cases} 1 & v \text{ real} \\ 2 & v \text{ complex} \end{cases}$

Dirichlet's unit thm: $\mathcal{O}_K^\times / \mu_K \xrightarrow{\sim}$ complete lattice in H.

$\Downarrow$ discrete

$\Rightarrow \mathcal{O}_K^\times \subset \mathop{\times}_{v \in S_\infty} K_v^\times$ discrete.

$\Rightarrow \mathcal{O}_K^\times \subset I_K^{S_\infty}$ discrete $\Rightarrow$ (a).

$T^1 := \{ (x_v)_{v \in S_\infty} \mid |\cdot| = 1 \} \subset \mathop{\times}_{v \in S_\infty} K_v^\times$

$$\begin{tikzcd} 1 \arrow[r] & \boxed{\text{compact}} \arrow[r] & T^1 \arrow[r] & \boxed{H} \arrow[r] & 0 \\ 1 \arrow[r] & \text{finite} \arrow[r] \arrow[u, phantom, "\cup"] & \mathcal{O}_K^\times \arrow[r] \arrow[u, phantom, "\cup"] & \text{complete lattice} \arrow[r] \arrow[u, phantom, "\cup"] & 0 \end{tikzcd}$$

$\Rightarrow T^1 / \mathcal{O}_K^\times$ compact.

Fix $v_0 \in S_\infty$, $\mathbb{R}^{>0} \hookrightarrow K_{v_0}^\times \subset I_K$

$\Rightarrow \mathbb{R}^{>0} \xrightarrow[|\cdot|]{\sim} \mathbb{R}^{>0}$

$\Rightarrow I_K^1 \times \mathbb{R}^{>0} \xrightarrow{\sim} I_K$

$C_K^1 \times \mathbb{R}^{>0} \xrightarrow{\sim} C_K$

$(I_k^{1} \cap I_k^{S_\infty}) / \mathcal{O}_k^\times$ compakt Hausdorff

$Cl(\mathcal{O}_k)$ endlich $\Longrightarrow$ $I_k^1 / K^\times$ kompakt Hausdorff.

$\underline{\underline{qed}}$

13.3 The reciprocity isomorphism

Now we return to L/K finite abelian with Galois group Γ.

Proposition 13.3.1: The norm map $I_L \to I_K$ induces a well-defined homomorphism

$L \otimes K_v = \times_{w|v} L_w$

$$\mathrm{Nm}_{L/K}\colon C_L \longrightarrow C_K.$$

Proof:

$$\begin{tikzcd}
C_L \arrow[r] & C_K \\
I_L \arrow[u] \arrow[r, "\mathrm{Nm}_{L/K}"] & I_K \arrow[u] \\
L^\times \arrow[u, phantom, "\cup"] \arrow[r, "\mathrm{Nm}_{L/K}"] & K^\times \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Lemma $\forall v \in \Omega_K$: $\forall x \in K$

$$\mathrm{Nm}_{L/K} x = \det_K(x\cdot : L \to L)$$
$$= \det_{K_v}(x\cdot : L \otimes_K K_v \to L \otimes_K K_v)$$
$$= \prod_{w|v} \det_{K_v}(x\cdot : L_w \to L_w)$$
$$= \prod_{w|v} \mathrm{Nm}_{L_w/K_v}(x)$$

$$\prod_v K_v^\times \twoheadrightarrow \bigoplus_v \Gamma_v$$

Big Theorem 13.3.2: The homomorphism $I_K \twoheadrightarrow \Gamma$ from 13.1.5 induces an isomorphism

$$\boxed{C_K/\,\mathrm{Nm}_{L/K}\, C_L \stackrel{\sim}{\longrightarrow} \Gamma.}$$

reciprocity isomorphism.

$$\begin{tikzcd}
I_K/\mathrm{Nm}_{L/K} I_L \arrow[r, two heads] & \Gamma \\
K^\times \arrow[u] &
\end{tikzcd}$$

Theorem 13.3.3: (a) The map $L \mapsto \mathcal{N}_L := \mathrm{Nm}_{L/K} C_L$ is a bijection from the set of finite abelian extensions of K up to isomorphism to the set of closed subgroups of finite index of C_K.

(b) We have

$$\begin{aligned} L_1 \subset L_2 &\iff \mathcal{N}_{L_1} \supset \mathcal{N}_{L_2}, \\ \mathcal{N}_{L_1 L_2} &= \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}, \quad \text{and} \\ \mathcal{N}_{L_1 \cap L_2} &= \mathcal{N}_{L_1} \mathcal{N}_{L_2}. \end{aligned}$$

Remark 13.3.4: The local behavior of an abelian extension L/K at a place $v \in M_K$ can be read off from the global information by pullback under the embedding

$$K_v^\times \hookrightarrow C_K, \quad x_v \mapsto [(1, \ldots, 1, x_v, 1, \ldots)].$$

Corollary 13.3.5: There is a natural isomorphism

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \;\cong\; (C_K)\hat{}. \quad := \varprojlim_{\substack{N \subset C_K \\ \text{open of finite index}}} C_K/N$$

$$K = \mathbb{Q} \Rightarrow Cl(\mathcal{O}_K) = 0$$

$$I_{\mathbb{Q}}^{S_\infty} = \mathbb{R}^\times \times \prod_p \mathbb{Z}_p^\times = \mathbb{R}^\times \times \hat{\mathbb{Z}}^\times$$

$$C_{\mathbb{Q}} \cong I_{\mathbb{Q}}^{S_\infty} / \mathcal{O}_K^\times = I_{\mathbb{Q}}^{S_\infty}/\{\pm 1\} \xleftarrow{\sim} \mathbb{R}^{>0} \times \hat{\mathbb{Z}}^\times$$

$$\Rightarrow (C_{\mathbb{Q}})\hat{} \cong \hat{\mathbb{Z}}^\times.$$

Remark 13.3.6: In the case $K = \mathbb{Q}$ the embedding $\hat{\mathbb{Z}}^\times \hookrightarrow I_{\mathbb{Q}}$ induces an isomorphism

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \;\cong\; (C_{\mathbb{Q}})\hat{} \;\cong\; \hat{\mathbb{Z}}^\times.$$

This isomorphism is the reciprocal (!) of that induced by the cyclotomic character:

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \;=\; \mathrm{Gal}\big(\mathbb{Q}(\textstyle\bigcup_n \mu_n)/\mathbb{Q}\big) \;\cong\; \hat{\mathbb{Z}}^\times.$$