Reminder:

We fix a number field K and let M_K denote the set of absolute values of K up to equivalence. Let S_∞ denote the subset of archimedean absolute values and $v \in M_K \smallsetminus S_\infty$ the respective normalized valuation.

Definition 13.1.2: The group of *ideles of K* is the subgroup

$$I_K \;:=\; \Big\{(x_v)_v \in \bigtimes_{v\in M_K} K_v^\times \;\Big|\; \forall' v\colon x_v \in \mathcal{O}_{K_v}^\times\Big\}.$$

It is endowed with the topology for which the subgroups

$$I_K^S \;:=\; \Big\{(x_v)_v \in \bigtimes_{v\in M_K} K_v^\times \;\Big|\; \forall v \notin S\colon x_v \in \mathcal{O}_{K_v}^\times\Big\} \;\cong\; \bigtimes_{v\in S} K_v^\times \times \bigtimes_{v\in M_K\smallsetminus S} \mathcal{O}_{K_v}^\times$$

for all finite subsets $S \subset M_K$ with $S_\infty \subset S$ are open and carry the product topology.

Definition 13.2.1: We call $C_K := I_K/K^\times$ the group of *idele classes*.

Proposition 13.2.2: There is a natural isomorphism $I_K/K^\times I_K^{S_\infty} \cong \mathrm{Cl}(\mathcal{O}_K)$.

reciprocity

Big Theorem 13.3.2: For any finite abelian extension L/K there is a natural isomorphism

$$C_K/\operatorname{Nm}_{L/K} C_L \xrightarrow{\sim} \operatorname{Gal}(L/K).$$

Theorem 13.3.3: The map $L \mapsto \mathcal{N}_L := \operatorname{Nm}_{L/K} C_L$ is a bijection from the set of finite abelian extensions of K up to isomorphism to the set of closed subgroups of finite index of C_K.

13.4 Class fields

$v \in S_\infty \rightsquigarrow$ no local condition

$v \notin S_\infty$, $v \nmid \mathfrak{m}$ $\rightsquigarrow$ no local factor of $I_K^\mathfrak{m}$ in $\mathcal{O}_{K_v}^\times$ $\Rightarrow H_\mathfrak{m}/K$ unramified at v.

Definition 13.4.1: For any non-zero ideal $\mathfrak{m} \subset \mathcal{O}_K$ we consider the open subgroup

$$I_K^\mathfrak{m} := \bigtimes_{v \in S_\infty} (K_v^\times)^\circ \times \bigtimes_{v \notin S_\infty} \{x_v \in \mathcal{O}_{K_v}^\times \mid x_v \equiv 1 \bmod \mathfrak{m}\mathcal{O}_{K_v}\} \;<\; I_K.$$

The finite abelian extension $H_\mathfrak{m}/K$ with $\cancel{\mathcal{N}_{H_\mathfrak{m}}} = \boxed{I_K/K^\times I_K^\mathfrak{m}}$ is called the *big ray class field of modulus* $\mathfrak{m}$ *of* K.

$\mathcal{N}_{H_\mathfrak{m}} = K^\times I_K^\mathfrak{m}/K^\times$

Theorem 13.4.2: Up to isomorphism $H_\mathfrak{m}/K$ is the unique maximal abelian extension L/K whose ramification at all finite places v satisfies

$$\{x_v \in \mathcal{O}_{K_v}^\times \mid x_v \equiv 1 \bmod \mathfrak{m}\mathcal{O}_{K_v}\} \;\subset\; \mathrm{Nm}_{L_v/K_v}\,\mathcal{O}_{L_v}^\times.$$

Definition 13.4.3: The extension $H := H_{(1)}$ is called the *big Hilbert class field of* K.

Theorem 13.4.4: Up to isomorphism H/K is the unique maximal abelian extension of K that is everywhere unramified.

Variant 13.4.5: Replacing the archimedean factors $(K_v^\times)^\circ$ by $K_v^\times$ in 13.4.1 one obtains the *small ray class field $H'_\mathfrak{m}$ of modulus $\mathfrak{m}$ of K.* This is the maximal abelian extension with the same ramification conditions at all finite primes and with the additional condition that all infinite primes are totally split.

Definition 13.4.6: The extension $H' := H'_{(1)}$ is called the *small Hilbert class field of K.*

Theorem 13.4.7: (a) Up to isomorphism H'/K is the unique maximal abelian extension of K that is everywhere unramified with all infinite primes totally split.

(b) There is a natural isomorphism $\mathrm{Gal}(H'/K) \cong \mathrm{Cl}(\mathcal{O}_K)$.

$\forall L/K$ fin:
$$\begin{array}{ccc} \mathrm{Frac}(\mathcal{O}_K) & \longrightarrow & \mathrm{Frac}(\mathcal{O}_L) \\ \mathfrak{a} & \longmapsto & \mathcal{O}_L \cdot \mathfrak{a} \\ \rightsquigarrow \mathrm{Cl}(\mathcal{O}_K) & \longrightarrow & \mathrm{Cl}(\mathcal{O}_L) \end{array}$$

Theorem 13.4.8: For every fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$ the ideal $\mathfrak{a}\mathcal{O}_{H'}$ is principal.

To rescue the properties of principal ideal domains for number fields one may try to pass to the Hilbert class field; but that may itself have a non-trivial class group. Even repeating the procedure does not solve the problem:

Theorem 13.4.9: *(Golod-Shafarevich 1963)* There exists a number field which does not possess a finite extension of class number 1.

$\mathbb{Z}_p$-extensions :

$$I_K^{S_\infty} = \prod_{v\in S_\infty} K_v^\times \times \prod_{v\notin S_\infty} \mathcal{O}_v^\times \subset I_K$$

in the kernel; $\mathcal{O}_v^\times \cong \text{finite} \times \mathbb{Z}_\ell^{[K_v/\mathbb{Q}_\ell]}$, in the kernel if $\ell \neq p$.

$$\mathrm{Gal}(L/K) \cong \mathbb{Z}_p \Rightarrow \mathbb{Z}_p \twoheadleftarrow C_K \twoheadleftarrow I_K$$

$$I_K / I_K^{S_\infty}\cdot K^\times = Cl(\mathcal{O}_K) \text{ finite.}$$

$\rightsquigarrow$ the maximal quotient of $I_K^{S_\infty}$ that is torsion free $\mathbb{Z}_p$-module

$$= \left(\text{quotient of } \prod_{v\mid p} \mathcal{O}_v^\times\right) = \prod_{v\mid p} \mathbb{Z}_p^{[K_v/\mathbb{Q}_p]} = \mathbb{Z}_p^{[K/\mathbb{Q}]}$$

$n := [K/\mathbb{Q}]$

$$\mathcal{O}_K^\times \cong \text{finite} \times \mathbb{Z}^{r+s-1}$$

$$U \cap \mathcal{O}_K^\times \cong \mathbb{Z}^{r+s-1}$$

$$C_K = I_K / K^\times \underset{\text{finite}}{\supset} I_K^{S_\infty}\cdot K^\times / K^\times \cong I_K^{S_\infty} / (I_K^{S_\infty} \cap K^\times) = I_K^{S_\infty} / \mathcal{O}_K^\times$$

Up to finite index want : $\boxed{\left(\prod_{v\mid p} \mathcal{O}_{K_v}^\times\right) / \overline{\mathcal{O}_K^\times}}$

$$\cup$$

$$U / \underbrace{\overline{U\cap \mathcal{O}_K^\times}}_{\text{closure of a subgroup} \,\cong\, \mathbb{Z}^{r+s-1}}$$

$$U := \prod_{v\mid p} (1 + p^2 \mathcal{O}_{K_v})$$

$$\mathbb{Z}_p^n \cong \bigoplus p^2 \cdot \mathcal{O}_{K_v} \quad (\exp / \log)$$

$$\mathbb{Z}_p^n$$

$$\mathbb{Z}_p^n / \overline{\log (U \cap \mathcal{O}_K^\times)} .$$

Let M/K be the union of all $\mathbb{Z}_p$-extensions of K. Then $\mathrm{Gal}(M/K) \cong \mathbb{Z}_p^m$ where $m = \mathrm{rank}_{\mathbb{Z}_p} \left(\mathbb{Z}_p^n / \overline{\log (U\cap \mathcal{O}_K^\times)}\right)$

$\underline{n=2}$: imaginary quadratic $\Rightarrow \mathcal{O}_K^\times$ finite $\Rightarrow m = n = 2$. $\qquad$ $n = r + 2s$

$\underline{n=1}$: cyclotomic $\mathbb{Z}_p$-extension.

$\underline{n=2}$: real quadratic $\Rightarrow U \cap \mathcal{O}_K^\times \cong \mathbb{Z} \Rightarrow m = 1$.

$\underline{r+s-1 = 1} \Rightarrow$

$\underline{\text{Leopoldt conjecture}}$: $\overline{\log(U \cap \mathcal{O}_K^\times)} \cong \mathbb{Z}_p^{r+s-1}$. $\iff m = n - (r+s-1)$
$= s+1$

$\underline{\text{Thm}}$: yes if $K/\mathbb{Q}$ abelian.

13.5 Reciprocity laws

The global reciprocity isomorphism can be viewed as a far reaching generalization of the quadratic reciprocity law.

Take odd prime $\ell \neq p$. Let $\ell^* := (-1)^{\frac{\ell-1}{2}} \cdot \ell \Rightarrow K := \mathbb{Q}(\sqrt{\ell^*}) \subset \mathbb{Q}(\mu_\ell)$ of degree 2 over $\mathbb{Q}$ unique.

$$\mathrm{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q}) \cong \mathbb{F}_\ell^\times$$

$$\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{F}_\ell^\times/(\mathbb{F}_\ell^\times)^2$$

$$\mathrm{Frob}_p \longmapsto [p]$$

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{\ell^*}] \\ \mathbb{Z}[\frac{1+\sqrt{\ell^*}}{2}] \end{cases}$$

$$\Rightarrow \mathcal{O}_K/p\mathcal{O}_K \cong \mathbb{Z}[X]/(X^2-\ell^*, p) \cong \mathbb{F}_p[X]/(X^2-\ell^*)$$

$\Rightarrow \left(\frac{\ell^*}{p}\right) = 1 \Leftrightarrow X^2-\ell^*$ splits over $\mathbb{F}_p \Leftrightarrow p$ splits in $K \Leftrightarrow [p] \in (\mathbb{F}_\ell^\times)^2 \Leftrightarrow \left(\frac{\ell}{p}\right) = 1$

<u>Explanation with CFT</u>

$$\mathrm{Gal}(K/\mathbb{Q}) = C_\mathbb{Q}/\mathrm{Nm}_{K/\mathbb{Q}} C_K = I_\mathbb{Q}/\mathbb{Q}^\times \cdot \mathrm{Nm}_{K/\mathbb{Q}} I_K$$

$$\mathrm{Frob}_p \longmapsto [(1, \dots, 1, p, 1, \dots)]$$

$$= [(p^{-1}, \dots, p^{-1}, 1, p^{-1}, \dots)]$$

at p

ℓ

At any prime $\neq p, \ell$ of $\mathbb{Z}$ the element p^{-1} is a local norm, because the prime is unramified in K.

At the infinite prime of $\mathbb{Q}$ " " " " " "

$$= \big[(1, \ldots, 1, \underbrace{p^{-1}}_{\text{at } \ell}, 1, \ldots) \big]$$

This class is the image of $[p^{-1}] \in \mathbb{Q}_\ell^\times / \mathrm{Nm}_{K_\ell/\mathbb{Q}_\ell} K_\ell^\times \cong \mathrm{Gal}(K_\ell/\mathbb{Q}_\ell)$ $\quad (*)$

$$K_\ell = \mathbb{Q}_\ell(\sqrt{\ell^*})$$

$\Rightarrow \mathrm{Nm}_{K_\ell/\mathbb{Q}_\ell} K_\ell^\times$ contains $\mathrm{Nm}_{K_\ell/\mathbb{Q}_\ell}(\sqrt{\ell^*}) = -\ell^*$

ℓ odd $\Rightarrow 1 + \ell \cdot \mathbb{Z}_\ell \subset \mathrm{Nm}_{K_\ell/\mathbb{Q}_\ell} K_\ell^\times$.

$$\Rightarrow \underbrace{(\mathrm{Nm}_{K_\ell/\mathbb{Q}_\ell} K_\ell^\times) \cap \mathbb{Z}_\ell^\times}_{\text{index } 2} = \underbrace{\mathrm{Nm}_{K_\ell/\mathbb{Q}_\ell} \mathcal{O}_{K_\ell}^\times}_{\supseteq (\mathcal{O}_{K_\ell}^\times)^2}$$

$$\Rightarrow \mathrm{Nm}_{K_\ell/\mathbb{Q}_\ell}(K_\ell^\times) = (-\ell^*)^{\mathbb{Z}} \cdot (\mathcal{O}_{K_\ell}^\times)^2.$$

So $\left(\frac{\ell^*}{p}\right) = 1 \iff p$ splits in $K \iff [(1, \ldots, 1, p, 1, \ldots)] = 1 \iff [p^{-1}] = 1$ in $(*)$

$\iff p^{-1} \in \mathrm{Nm}_{K_\ell/\mathbb{Q}_\ell} K_\ell^\times \iff p^{-1}$ square in $\mathbb{F}_\ell^\times \iff \left(\frac{p}{\ell}\right) = 1$.