# Exam questions

**Exercise 1.** For $K$ an algebraically closed field of characteristic not equal to $2, 3$ and $a, b \in K$ consider the plane cubic curve

$$C = \{(x, y) \in \mathbb{A}^2 : y^2 = x^3 + ax + b\} \subseteq \mathbb{A}^2.$$

Prove a criterion in terms of $a, b$ when the curve $C$ is smooth.

**Exercise 2.** Consider the rational map

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, (X : Y : Z) \mapsto (1/X : 1/Y : 1/Z).$$

Find the largest open subset $U \subset \mathbb{P}^2$ such that some representative of the rational map $f$ is defined on $U$ (i.e. the largest $U$ to which $f$ can be extended). Show that $f$ is birational and describe its inverse.

**Exercise 3.** Let $X = V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subseteq \mathbb{P}^3_{\mathbb{C}}$. Write down a non-constant morphism $f : \mathbb{P}^1_{\mathbb{C}} \to X$. How many such morphisms are there?

**Exercise 4.** Show that for any ring $R$ there exists a unique morphism

$$X = \text{Spec}(R) \to \text{Spec}(\mathbb{Z}).$$

For $\mathbb{F}_2$ the finite field with two elements, let $X = \text{Spec}(\mathbb{F}_2)$ and $Y = V(x^2 + 1) \subseteq \mathbb{A}^1_{\mathbb{Z}}$. How many points does (the underlying topological space of) the fiber product $X \times_{\text{Spec}(\mathbb{Z})} Y$ have?

**Exercise 5.** Let $K$ be an algebraically closed field. For the following combinations of properties of schemes or morphisms, can you either give an example (with proof) or show that they cannot exist?

*a)* an affine variety of dimension 1 over $K$ which is complete

*b)* a morphism $f : \mathbb{P}^1 \to \mathbb{P}^2$ passing through the points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$

*c)* a scheme $X$ which is not reduced, but which contains a non-empty open subscheme $U \subseteq X$ which is reduced

**Exercise 6.** Let $K$ be an algebraically closed field. For the following combinations of properties of schemes or morphisms, can you either give an example (with proof) or show that they cannot exist?

*a*) a reduced and separated scheme $X$ over $K$ which is not a variety over $K$

*b*) a surjective map $f : X \to Y$ of projective varieties with $\dim X = 2$ and $\dim Y = 1$

*c*) a birational map $f : X \to Y$ between irreducible projective varieties $X, Y$ over $K$ which is not surjective

**Exercise 7.** Consider the map $f : \mathbb{A}^1 \to \mathbb{A}^3, t \mapsto (t, t^2, t^3)$ of affine varieties over an algebraically closed field $K$.

*a*) Show that $f$ is a closed morphism and an isomorphism onto its image $Y = f(\mathbb{A}^1) \subseteq \mathbb{A}^3$.

*b*) For $U = \mathbb{A}^3 \setminus Y$, what is $\mathcal{O}_U(U)$?

**Exercise 8.** Let $f : X \to Y$ be a closed morphism of irreducible varieties over an algebraically closed field. Show that if $f$ is surjective, then $\dim X \geq \dim Y$.

**Exercise 9.** For the Grassmannian $G(k, n)$, show that the incidence correspondence

$$I = \{(L, a) \in G(k, n) \times \mathbb{A}^n : a \in L\}$$

is a closed subvariety of $G(k, n) \times \mathbb{A}^n$ and compute its dimension.

**Exercise 10.**

*a*) Write down (with proof) a locally free sheaf $\mathcal{F}$ of rank 2 on $\mathbb{P}^n$ whose space $\mathcal{F}(\mathbb{P}^n)$ of global sections contains only the zero section.

*b*) Is there a locally free sheaf $\mathcal{G}$ of rank 2 on $\mathbb{A}^2$ whose only global section is the zero section?