

## Exam questions

**Exercise 1.** For  $K$  an algebraically closed field of characteristic not equal to 2, 3 and  $a, b \in K$  consider the plane cubic curve

$$C = \{(x, y) \in \mathbb{A}^2 : y^2 = x^3 + ax + b\} \subseteq \mathbb{A}^2.$$

Prove a criterion in terms of  $a, b$  when the curve  $C$  is smooth.

**Exercise 2.** Consider the rational map

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, (X : Y : Z) \mapsto (1/X : 1/Y : 1/Z).$$

Find the largest open subset  $U \subset \mathbb{P}^2$  such that some representative of the rational map  $f$  is defined on  $U$  (i.e. the largest  $U$  to which  $f$  can be extended). Show that  $f$  is birational and describe its inverse.

**Exercise 3.** Let  $X = V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subseteq \mathbb{P}_{\mathbb{C}}^3$ . Write down a non-constant morphism  $f : \mathbb{P}_{\mathbb{C}}^1 \rightarrow X$ . How many such morphisms are there?

**Exercise 4.** Show that for any ring  $R$  there exists a unique morphism

$$X = \operatorname{Spec}(R) \rightarrow \operatorname{Spec}(\mathbb{Z}).$$

For  $\mathbb{F}_2$  the finite field with two elements, let  $X = \operatorname{Spec}(\mathbb{F}_2)$  and  $Y = V(x^2 + 1) \subseteq \mathbb{A}_{\mathbb{Z}}^1$ . How many points does (the underlying topological space of) the fiber product  $X \times_{\operatorname{Spec}(\mathbb{Z})} Y$  have?

**Exercise 5.** Let  $K$  be an algebraically closed field. For the following combinations of properties of schemes or morphisms, can you either give an example (with proof) or show that they cannot exist?

- a) an affine variety of dimension 1 over  $K$  which is complete
- b) a morphism  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  passing through the points  $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$
- c) a scheme  $X$  which is not reduced, but which contains a non-empty open subscheme  $U \subseteq X$  which is reduced

**Exercise 6.** Let  $K$  be an algebraically closed field. For the following combinations of properties of schemes or morphisms, can you either give an example (with proof) or show that they cannot exist?

- a) a reduced and separated scheme  $X$  over  $K$  which is not a variety over  $K$
- b) a surjective morphism  $f : X \rightarrow Y$  of projective varieties with  $\dim X = 2$  and  $\dim Y = 1$
- c) a birational morphism  $f : X \rightarrow Y$  between irreducible projective varieties  $X, Y$  over  $K$  which is not surjective

**Exercise 7.** Consider the map  $f : \mathbb{A}^1 \rightarrow \mathbb{A}^3, t \mapsto (t, t^2, t^3)$  of affine varieties over an algebraically closed field  $K$ .

- a) Show that  $f$  is a closed morphism and an isomorphism onto its image  $Y = f(\mathbb{A}^1) \subseteq \mathbb{A}^3$ .
- b) For  $U = \mathbb{A}^3 \setminus Y$ , what is  $\mathcal{O}_U(U)$ ?

**Exercise 8.** Let  $f : X \rightarrow Y$  be a closed morphism of irreducible varieties over an algebraically closed field. Show that if  $f$  is surjective, then  $\dim X \geq \dim Y$ .

**Exercise 9.** For the Grassmannian  $G(k, n)$ , show that the incidence correspondence

$$I = \{(L, a) \in G(k, n) \times \mathbb{A}^n : a \in L\}$$

is a closed subvariety of  $G(k, n) \times \mathbb{A}^n$  and compute its dimension.

**Exercise 10.**

- a) Write down (with proof) a locally free sheaf  $\mathcal{F}$  of rank 2 on  $\mathbb{P}^n$  whose space  $\mathcal{F}(\mathbb{P}^n)$  of global sections contains only the zero section.
- b) Is there a locally free sheaf  $\mathcal{G}$  of rank 2 on  $\mathbb{A}^2$  whose only global section is the zero section?