Presence Sheet 1

Exercise 1. The space $\mathrm{Mat}(m \times n) = \mathrm{Mat}(m \times n, K)$ of $m \times n$ matrices over the algebraically closed field K can be identified with the affine space $\mathbb{A}^{m \cdot n} = \mathbb{A}_K^{m \cdot n}$. Write $I_n \in \mathrm{Mat}(n \times n)$ for the identity matrix.

For the following subsets of various matrix spaces, decide whether they are Zariski open, closed, or neither.

a) $\mathrm{GL}(n) = \{A : A \text{ invertible}\} \subset \mathrm{Mat}(n \times n)$ general linear group

b) $\mathrm{SL}(n) = \{A : \det A = 1\} \subset \mathrm{Mat}(n \times n)$ special linear group

c) $\mathrm{Mat}^{\leq k}(m \times n) = \{A : \mathrm{rank} A \leq k\} \subset \mathrm{Mat}(m \times n)$ matrices of rank at most k

d) $\mathrm{Mat}^{\geq k}(m \times n) = \{A : \mathrm{rank} A \geq k\} \subset \mathrm{Mat}(m \times n)$ matrices of rank at least k

e) $U(n, \mathbb{C}) = \{A : A \cdot A^* = I_n\} \subset \mathrm{Mat}(n \times n, \mathbb{C})$ unitary matrices over $K = \mathbb{C}$

f) $\mathrm{Diag}(n) = \{A : A \text{ diagonal}\} \subset \mathrm{Mat}(n \times n)$ diagonal matrices

g) $\mathrm{Nil}(n) = \{A : A^m = 0 \text{ for some } m\} \subset \mathrm{Mat}(n \times n)$ nilpotent matrices

h) $\mathrm{Comm}(n) = \{(A, B) : A \cdot B = B \cdot A\} \subset \mathrm{Mat}(n \times n)^2$ pairs of commuting matrices

Exercise 2. Consider the set

$$\mathrm{Id}(2, \mathbb{C}) = \{A \in \mathrm{Mat}(2 \times 2, \mathbb{C}) : A^2 = I_2\}$$

of idempotent matrices. Show that it can be written as the disjoint union of 3 non-empty affine varieties. (*Bonus*: If you have seen the definition of irreducible decomposition: compute it for $\mathrm{Id}(2, \mathbb{C})$.)
Hint: You can find the components by solving equations, or by using results from Linear Algebra concerning eigenvalues, etc ...