Presence Sheet 10

Exercise 1. (Basic schemes)

a) For the zero ring $R = \{0\}$ show that $\operatorname{Spec} R = \emptyset$.

b) For a field K, show that $\operatorname{Spec} K$ has a unique point. Are fields the only rings with this property?

c) For any ring R and ideal I, consider the quotient map $\varphi : R \to R/I$. Show that the map

$$\Phi : \operatorname{Spec} R/I \to \operatorname{Spec} R, q \mapsto \varphi^{-1}(q)$$

is injective with image $V(I)$. Moreover, show that the pullback of the Zariski topology is the Zariski topology.
Note: This means that φ induces a homeomorphism from $\operatorname{Spec} R/I$ to $V(I) \subseteq \operatorname{Spec} R$.

d) For K a field and $m \in \mathbb{N}_{>0}$, what is the spectrum $\operatorname{Spec} K[x]/\langle x^m \rangle$ as a topological space?

Bonus exercise:

e) What is the spectrum $\operatorname{Spec} K[[t]]$ of the formal power series ring $K[[t]]$ in a single variable over a field K, as a topological space?

Exercise 2. (Zariski topology) Let R be a ring.

a) Show that for $p \subseteq R$ a prime ideal, the vanishing set $V(p)$ is irreducible.
Hint: There is a one-line argument using [Gathmann, Remark 12.9 (b)].

b) Let S be a reduced ring with $\operatorname{Spec}(S)$ irreducible. Show that S is an integral domain.
Hint: Remember that in any ring, the nilradical $\sqrt{\langle 0 \rangle}$ is given by the intersection of all prime ideals of the ring.

c) Show that the irreducible closed subsets of $\operatorname{Spec} R$ are exactly given by $V(p)$ for $p \subseteq R$ a prime ideal.
Hint: For $V(J) \subseteq \operatorname{Spec}(R)$ closed, show that the map $\Phi : \operatorname{Spec}(R/\sqrt{J}) \to V(J)$ from Exercise 1 is a homeomorphism.

d) Conclude that $\dim \operatorname{Spec} R$ is given by the Krull dimension of R.