Presence Sheet 11

Exercise 1. (Tangent spaces) For two schemes X and Y over a base scheme S define the set of Y-points of X to be $\mathrm{Mor}_S(Y, X)$ and denote it by $X(Y)$. For a point x of a locally ringed space X the cotangent space of X at x is the $K(x)$–vector space

$$T^*_{X,x} := \mathrm{m}(x)/\mathrm{m}(x)^2\,,$$

where $\mathrm{m}(x) \subseteq \mathcal{O}_{X,x}$ is the maximal ideal. The tangent space of X at x is the $K(x)$-dual of $T^*_{X,x}$ and is denoted by $T_{X,x}$.

Suppose X is a variety over an algebraically closed field K. Consider schemes over $\mathrm{Spec}\, K$.

a) Show that $X(\mathrm{Spec}\, K)$ is naturally identified with the set of closed points of X. Is this true if K is not necessarily algebraically closed?

b) Show that any element of $X(\mathrm{Spec}\, K[t]/t^2)$ is naturally identified with a choice of a closed point $x \in X$ and a choice of an element $v \in T_{X,x}$.

c) The homomorphisms $K \hookrightarrow K[t]/t^2 \twoheadrightarrow K$ (where $t \mapsto 0$ in the second homomorphism) correspond to morphisms of schemes $\mathrm{Spec}\, K \to \mathrm{Spec}\, K[t]/t^2 \to \mathrm{Spec}\, K$. Describe the induced maps of sets $X(\mathrm{Spec}\, K) \to X(\mathrm{Spec}\, K[t]/t^2) \to X(\mathrm{Spec}\, K)$ in terms of the data above.

d) Let $f\colon X \to Y$ be a morphism of varieties over K and $x \in X$ be a closed point. Construct a natural linear morphism $df_x\colon T_{X,x} \to T_{Y,f(x)}$. You can use the definition of tangent space or the description from b).

Let Z be the fibre of f above $f(x)$. Show that $\mathrm{Ker}(df_x) \simeq T_{Z,x}$.

e) Consider $X = \mathbb{A}^n_K = \mathrm{Spec}\, K[x_1, \cdots x_n]$ and $o = (0, \cdots, 0) \in X$. Show that the cotangent space $T^*_{X,o}$ is identified with the standard n-dimensional space K^n via $\mathrm{m}(o) \ni x_i \mapsto e_i$. By taking the dual basis we get a similar identification of $T_{X,o}$ with K^n. Now let $a = (a_1, \cdots, a_n)$ be an arbitrary closed point of A^n_K. By the translation morphism $p \mapsto p + a$ of $\mathbb{A}^n_K$ identify $T_{X,a}$ with $T_{X,o} \simeq K^n$ using d).

f) Let $X := \mathbb{A}^{n+1}_K \backslash \{(0, 0, \cdots, 0)\} \to \mathbb{P}^n_K$ be the natural morphism. Recall that geometrically it maps a closed point $a = (a_0, \cdots, a_n)$ to the line ℓ_a through o and a. Using e) identify any tangent space at any closed point of $X \subset \mathbb{A}^{n+1}_K$ with the standard K^{n+1}. Show that $T_{X,a} \to T_{\mathbb{P}^n_K, \ell_a}$ is surjective and describe $T_{\mathbb{P}^n_K, \ell_p}$ as a quotient of K^{n+1}.