Presence Sheet 12

Exercise 1. (Subschemes) Are the following morphisms examples of open subschemes, closed subschemes or neither?

a) $f : X = \operatorname{Spec} K[x,y]/(xy-1) \to \operatorname{Spec} K[x] = Y$ unique map with

$$f^* : K[x] \to K[x,y]/(xy-1), x \mapsto x$$

b) $f : \mathbb{P}^1 \to \mathbb{P}^2, (x_0 : x_1) \mapsto (x_0 : x_1 : 0)$

c) $f : \mathbb{A}^1 \to \mathbb{A}^2, t \mapsto (t^2, t^3)$

Exercise 2. (Fiber products)

a) Let X, S be schemes, then the set of *S-points of X* is given by

$$X(S) = \{S \xrightarrow{f} X : f \text{ morphism}\}.$$

For $g : X \to Y$ a morphism of schemes, there is a natural map

$$X(S) \to Y(S), f \mapsto g \circ f$$

of their S-points.

Show that for $f_X : X \to Z$ and $f_Y : Y \to Z$ two morphisms of schemes, we have

$$(X \times_Z Y)(S) = \{(x,y) \in X(S) \times Y(S) : f_X \circ x = f_Y \circ y \in Z(S)\}.$$

Note: This makes precise the idea that points of $X \times_Z Y$ are pairs of points of X, Y mapping to the same point in Z.

b) Calculate the following fiber products $X \times_Z Y$:

i) $X = \mathbb{A}^1 \xrightarrow{x \mapsto (x,0)} \mathbb{A}^2 = Z$ and $X = \mathbb{A}^1 \xrightarrow{y \mapsto (0,y)} \mathbb{A}^2 = Z$

ii) $X = \mathbb{A}^1 \xrightarrow{t \mapsto (t,t)} \mathbb{A}^2 = Z$ and $X = \mathbb{A}^3 \xrightarrow{(x,y,z) \mapsto (x,y)} \mathbb{A}^2 = Z$

Hint: Remember that for $\varphi : S \to R$ a ring homomorphism and $I \subseteq S$ an ideal we have $R \otimes_S (S/I) \cong R/\langle \varphi(I) \rangle$.

c) Show that for a fiber product

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_Y} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{f_X} & Z \end{array}$$

such that f_X is a closed subscheme, also π_Y is a closed subscheme.

d) Is it true that for X, Y, Z varieties also $X \times_Z Y$ is a variety?