Presence Sheet 13

In the sheet below we discuss sheaves on topological spaces X which are not schemes (and sometimes sheaves which are not $\mathcal{O}_X$-modules). However, all definitions relevant for the exercises below (stalks, sheafification, injective and surjective morphisms of sheaves, etc) still make sense, even though we presented them for schemes.

Exercise 1. (Sheaffification of the constant presheaf) Let X be a topological space and S a set. Let $\mathcal{F}$ be the presheaf (of sets) given by

$$\mathcal{F}(U) = \{f : U \to S : f \text{ constant}\} \text{ for } U \subseteq X \text{ open}\,.$$

a) Show that the stalk $\mathcal{F}_p$ at any $p \in X$ is isomorphic to S.

b) Define the sheaf $\underline{S}$ of locally constant functions to S.
Hint: If you're unsure what "locally constant" means, ask wikipedia, ChatGPT or the lecturer.

c) Show that $\underline{S}$ is the sheafification of $\mathcal{F}$.
Hint: You can use the definition directly or prove that $\mathcal{G}$ is a sheaf and that the natural map $\mathcal{F} \to \underline{S}$ is an isomorphism on stalks.

Exercise 2. (Exponential exact sequence reloaded) Recall from Presence sheet 3 that for the complex numbers $X = \mathbb{C}$ with the Euclidean topology and $U \subseteq \mathbb{C}$ open, we have two sheaves

$$\mathcal{O}^{\text{hol}}(U) = \{f : U \to \mathbb{C} : f \text{ holomorphic}\},$$
$$\mathcal{O}^{\text{hol},\times}(U) = \{f : U \to \mathbb{C}^* : f \text{ holomorphic}\}\,.$$

on X and the exponential map $\exp : \mathcal{O}^{\text{hol}} \to \mathcal{O}^{\text{hol},\times}$ given by

$$\exp_U : \mathcal{O}^{\text{hol}}(U) \to \mathcal{O}^{\text{hol},\times}(U), f \mapsto \exp(f)\,.$$

a) Show that exp is a surjective map of sheaves (of abelian groups).
Bonus question: Is $\exp_U$ surjective for all $U \subseteq X$ open?

b) What is the kernel sheaf of exp?
Hint: Exercise 1.

c) With the two previous exercise parts in mind, what do you think is the exponential exact sequence?