Presence Sheet 2

Exercise 1. Show that a map $F : \mathbb{A}^n \to \mathbb{A}^m$ of the form $F(x) = (f_1(x), \ldots, f_m(x))$ with $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ is continuous (with respect to the Zariski topology).
Note: Such polynomial maps will be examples of morphisms of affine varieties later, but for the purpose of the exercises below we just need their continuity.

Exercise 2. Given $d \in \mathbb{N}$, we can identify the set $P_d \subseteq K[x]$ of monic degree d polynomials in with $\mathbb{A}^d$ by the map

$$\mathbb{A}^d \xrightarrow{\sim} P_d, (a_0, \ldots, a_{n-1}) \mapsto x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0 \,.$$

a) Show that the set $\Delta_2 \subseteq P_2$ of degree 2 polynomials with a double zero is a Zariski closed subset.

b) Show that the set $\Delta_d \subseteq P_d$ of degree d polynomials with a double zero is a hypersurface.
Hint: If you are stuck, you can try to google the word "discriminant".

c) Write down a polynomial map $F : \mathbb{A}^{d-1} \to P_d$ whose image is Δ_d, and conclude that Δ_d is irreducible.

Exercise 3. (Cayley-Hamilton theorem)
In this exercise we show the Cayley-Hamilton theorem from linear algebra. It says that for $A \in \mathrm{Mat}(n \times n, K)$ with characteristic polynomial $\chi_A(x) = \det(xE_n - A)$ we have $\chi_A(A) = 0$.

a) Show (or convince yourself) that the maps

$$\begin{aligned} \chi : &\quad \mathrm{Mat}(n \times n, K) \to P_n, A \mapsto \chi_A \\ \mathrm{ev} : &\quad \mathrm{Mat}(n \times n, K) \times P_m \to \mathrm{Mat}(n \times n, K), (A, f) \mapsto f(A) \end{aligned}$$

are polynomial maps in the sense of Exercise 1.

b) Show that the set $U = \{A \in \mathrm{Mat}(n \times n, K) : A \text{ has } n \text{ distinct eigenvalues}\}$ is a non-empty irreducible Zariski open subset of $\mathrm{Mat}(n \times n, K)$.

c) Prove the Cayley-Hamilton theorem for $A \in U$.
Hint: Note that for $S \in \mathrm{GL}(n, K)$ and $A \in \mathrm{Mat}(n \times n, K)$ we have $f(SAS^{-1}) = Sf(A)S^{-1}$ for any $f \in K[x]$.

d) Conclude that the Cayley-Hamilton theorem holds for all $A \in \mathrm{Mat}(n \times n, K)$.

Exercise 4. Compute the dimension of the sets

$$T = \{A \in \mathrm{Mat}(2 \times 2, K) : \mathrm{trace}(A) = 0\}, \mathrm{Nil}_2 = \{A \in \mathrm{Mat}(2 \times 2, K) : A \text{ nilpotent}\} \,.$$