Presence Sheet 3

Note: Since the Presence Sheet 2 was pretty long, the beginning of the class will discuss the remaining exercises there. Below we just have some short exercises on the material from week 3.

Exercise 1. For the complex numbers $X = \mathbb{C}$ with the Euclidean topology and $U \subseteq \mathbb{C}$ open, let

$$\mathcal{O}^{\text{hol}}(U) = \{f : U \to \mathbb{C} : f \text{ holomorphic}\},$$
$$\mathcal{O}^{\text{hol},\times}(U) = \{f : U \to \mathbb{C}^* : f \text{ holomorphic}\}.$$

a) Convince yourself that $\mathcal{O}^{\text{hol}}$ and $\mathcal{O}^{\text{hol},\times}$ are sheaves of abelian groups (with respect to pointwise addition and multiplication).

b) Show that the maps

$$\exp_U : \mathcal{O}^{\text{hol}}(U) \to \mathcal{O}^{\text{hol},\times}(U), f \mapsto \exp(f)$$

are well-defined group homomorphisms that are compatible under restriction maps. *Note:* Such a collection of maps gives rise to a *morphism of sheaves*

$$\exp : \mathcal{O}^{\text{hol}} \to \mathcal{O}^{\text{hol},\times}.$$

Show that exp induces a well-define map $\exp_0 : \mathcal{O}_0^{\text{hol}} \to \mathcal{O}_0^{\text{hol},\times}$ of the stalks at $0 \in \mathbb{C}$.

c) Show the the map

$$T : \mathcal{O}_0^{\text{hol}} \to \mathbb{C}[[z]], f \mapsto \sum_{k \geq 0} \frac{f^{(k)}(0)}{k!}(0) \cdot z^k$$

to the formal power series ring $\mathbb{C}[[z]]$ is well-defined and injective. What is its image?

Bonus questions for (complex) analysis enthusiasts:

d) Are the group homomorphisms $\exp_U$ injective (resp. surjective) for the unit disc $U = B_1(0)$ (resp. $U \subseteq \mathbb{C}$ any open subset)?

e) Compute the kernel and the image of the map $\exp_0$ between the stalks at 0. Use this to calculate the stalk $\mathcal{O}_0^{\text{hol},\times}$.

f) On $X = \mathbb{R}$ with the Euclidean topology consider the sheaf $\mathcal{C}^\infty$ of infinitely differentiable (or smooth) functions to $\mathbb{R}$. Is the analogous Taylor expansion map $T : \mathcal{C}_0^\infty \to \mathbb{R}[[x]]$ still injective?