Presence Sheet 4

Exercise 1. A *linear algebraic group* is a tuple (G, m, i, e) of an affine variety G, morphisms

$$m : G \times G \to G \text{ and } i : G \to G$$

and a point $e \in G$ such that

$$\begin{aligned} m(m(g_1, g_2), g_3) &= m(g_1, m(g_2, g_3)) \\ m(e, g) &= m(g, e) = g \\ m(g, i(g)) &= m(i(g), g) = e \end{aligned}$$

for all $g, g_1, g_2, g_3 \in G$. We think of $m(g, h) = g \circ h$ as the group operation, $e \in G$ as the neutral element of the group and $i(g) = g^{-1}$ as the inverse element in the group.

Show that the following are examples of linear algebraic groups (provide the full data (G, m, i, e) above, show that m, i are morphisms and check as many of the properties as you find interesting):

a) $\mathbb{G}_a = \mathbb{A}^1$ with addition $+$

b) $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ with multiplication $\cdot$

c) $\mu_2 = \{1, -1\}$ with multiplication $\cdot$

d) $\mathrm{GL}_n = \{A \in \mathrm{Mat}(n \times n, K) : A \text{ invertible}\}$ with matrix multiplication
Hint: If you are stuck, you can look up the "adjugate matrix" on wikipedia.

Exercise 2. In this exercise, we want to show the following nice topological property of linear algebraic groups:

Proposition Any connected linear algebraic group G is irreducible.

a) Let X, Y be affine varieties and $y_0 \in Y$. Show that the constant map $X \to Y, x \mapsto y_0$ is a morphism.
Bonus challenge: Show the same thing for X, Y prevarieties!

b) Show that for $h \in G$ the *left-translation*

$$t_h : G \to G, g \mapsto m(h, g)$$

is an isomorphism.

c) Show that for any two points $p, q \in G$ there is an isomorphism $\varphi : G \to G$ with $\varphi(p) = q$.

d) Let X be a connected topological space with irreducible decomposition $X = X_1 \cup \ldots \cup X_n$ with $n \geq 2$. Show that there exist

- a point $p \in X$ lying on a unique (i.e. exactly one) irreducible component X_i,
- a point $q \in X$ lying on at least two irreducible components

e) Prove the proposition above.