Presence Sheet 5

Exercise 1. Let $0 \neq f \in K[x_0, x_1]$ be a nonzero homogeneous polynomial of degree $d \geq 0$.

a) Let $g(x_1) = f(1, x_1)$ be the dehomogenization of f. Show that

$$f(x_0, x_1) = x_0^d \cdot g(x_1/x_0).$$

b) Show that f has a decomposition

$$f = (b_1 x_0 - a_1 x_1) \cdot (b_2 x_0 - a_2 x_1) \cdots (b_d x_0 - a_d x_1)$$

into linear factors.

c) Show that the vanishing set of f is given by

$$V(f) = \{(a_1 : b_1), \ldots, (a_d : b_d)\} \subseteq \mathbb{P}^1 .$$

We say that the $(a_i : b_i)$, counted with multiplicity, are the zeros of f on $\mathbb{P}^1$.
Note: These multiplicities sum to the degree d of the polynomial.

Exercise 2. A homogeneous polynomial $f \in K[x_0, x_1]$ of degree $d \geq 0$ is given by

$$f = f_c = c_0 x_0^d + c_1 x_0^{d-1} x_1 + \ldots + c_{d-1} x_0 x_1^{d-1} + c_d x_1^d .$$

In the following we identify the space Poly_d of such nonzero polynomials up to scaling with $\mathbb{P}^d$ by sending the class $[f_c]$ of the polynomial f_c to the vector $c = (c_0 : c_1 : \ldots : c_d) \in \mathbb{P}^d$.

For the following sets, decide if they are open, closed or not-well-defined in $\mathrm{Poly}_d = \mathbb{P}^d$, and in the first two cases compute their dimension (assume $d \geq 1$ for simplicity).

a) $A = \{[f] \in \mathrm{Poly}_d : f(p_0) = 0 \text{ for } p_0 = (1 : 0) \in \mathbb{P}^1\}$

b) $B = \{[f] \in \mathrm{Poly}_d : f(p_0) = f(p_1) \text{ for } p_0 = (1 : 0), p_1 = (0 : 1) \in \mathbb{P}^1\}$

c) $C = \{[f] \in \mathrm{Poly}_d : \text{all zeros of } f \text{ have multiplicity } 1\}$

Bonus exercise (optional; guess an answer - proof needs tools we'll not discuss):

d) $D = \{[f] \in \mathrm{Poly}_d : f \text{ has a zero of order at least } 3\}$

Exercise 3. An *effective divisor of degree d* on $\mathbb{P}^1$ is a formal linear combination $D = m_1 p_1 + \ldots + m_k p_k$ of finitely many points $p_i \in \mathbb{P}^1$ with $m_i \in \mathbb{N}$ such that $m_1 + \ldots + m_k = d$. E.g. examples of effective divisors of degree 3 are:

$$D_1 = (0 : 1) + (1 : 1) + (1 : 0) \text{ and } D_2 = 2 \cdot (1 : 2) + (1 : 3) . \quad (1)$$

Let Eff_d be the set of such effective divisors.

a) Show that the map

$$\Psi : \mathrm{Eff}_d \to \mathrm{Poly}_d \cong \mathbb{P}^d, D = \sum_{i=1}^{k} m_i(a_i : b_i) \mapsto \left[\prod_{i=1}^{k} (b_i x_0 - a_i x_1)^{m_i}\right] \tag{2}$$

is well-defined and bijective. Thus we can interpret $\mathbb{P}^d$ as the set of effective divisors of degree d on $\mathbb{P}^1$.

b) What are the images of D_1, D_2 from (1) under Ψ?