Presence Sheet 7

On this exercise sheet, we'll talk a bit about the topology of algebraic varieties (and work over the field $K = \mathbb{C}$ of the complex numbers everywhere). If X is a topological space, its *Euler characteristic* is defined as the alternating sum

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}} H_i(X, \mathbb{Q})$$

of the dimensions of its homology groups.

However, even without knowing anything about homology groups, you can do the entire sheet below just using the following three properties of the Euler characteristic of (complex) algebraic varieties:

(A) If X is a variety which can be written as a disjoint union $X = X_1 \sqcup \ldots \sqcup X_m$ of finitely many locally closed[1] sets $X_i \subseteq X$, then $\chi(X) = \chi(X_1) + \ldots + \chi(X_m)$.

(B) If $\pi : X \to Y$ is a morphism, such that all fibers $X_q = \pi^{-1}(q)$ have the same Euler characteristic $\chi(X_q) = d$ for $q \in Y$, then $\chi(X) = d \cdot \chi(Y)$.

(C) We have $\chi(\{pt\}) = \chi(\mathbb{A}^1) = 1$.

Exercise 1. (Basic spaces)

a) Calculate $\chi(\mathbb{P}^1)$ and $\chi(\mathbb{A}^1 \setminus \{0\})$.

b) Show that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$ for X, Y algebraic varieties.

c) Calculate $\chi(\mathbb{A}^n)$.

d) Calculate $\chi(\mathbb{P}^n)$.

Hint: Start with $n = 2$ in c) and d) if you are stuck.

Exercise 2. (Fancier spaces)

a) Calculate $\chi(X)$ for $X = V(x_1x_2) \subseteq \mathbb{A}^2$.

b) Let $Q_n \subseteq \mathbb{P}^n$ denote an irreducible quadric hypersurface. Calculate $\chi(Q_2)$ for *all* such hypersurfaces and $\chi(Q_3)$ for *one* such hypersurface. What happens for a *reducible* quadric hypersurface in $\mathbb{P}^2$?

[1] Recall that a set $S \subseteq X$ is locally closed if it can be written as the intersection $S = U \cap C$ of a Zariski open set $U \subseteq X$ and a Zariski closed set $C \subseteq X$.

Exercise 3. (Fanciest spaces)

a) Compute the Euler characteristic $\chi(G(2,4))$.
Hint: Look at [Gathmann, Remark 8.20].
Bonus: Can you find the formula for $\chi(G(k,n))$?

b) Let $0 \neq f \in K[x_1]$ be a homogeneous polynomial of degree $2g+2$ for some $g \in \mathbb{N}$ which has only simple zeros. Consider the affine curve

$$C = \{(x_1, x_2) \in \mathbb{A}^2 : x_2^2 = f(x_1)\} \subseteq \mathbb{A}^2$$

The variety C is called an *affine hyperelliptic curve of genus g*. Show that $\chi(C) = -2g$.
Hint: Try to find a morphism from C to a simpler space, which has finite fibers.