Presence Sheet 8

Exercise 1. (Rational maps $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$)

a) Let $f_1, f_2 \in K[x_0, x_1]$ be two non-zero homogeneous polynomials of the same degree $d \geq 0$. Show that the assignment

$$f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1, x \mapsto (f_1(x) : f_2(x)) \tag{1}$$

gives a rational map. What is the open set of $\mathbb{P}^1$ where it is defined?

b) Compute the domain of definition of f for $f_1 = x_0^2 + x_0x_1$ and $f_2 = x_0x_1 + x_1^2$. Show that on its domain of definition, f is given by the identity function (and thus can be extended to all of $\mathbb{P}^1$).

c) Use the idea of the last part to show that any map f as in (1) can be extended to all of $\mathbb{P}^1$.
Hint: It might help to do an induction on the degree d.

d) Let $g \in K[x]$ be a non-constant polynomial with associated distinguished open $D(g) \subseteq \mathbb{A}^1$ and $\widetilde{f} : D(g) \to \mathbb{A}^1$ a morphism. Seeing $D(g) \subseteq \mathbb{A}^1 \subseteq \mathbb{P}^1$ show that there is a rational map $f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ of the form (1) that agrees with $\widetilde{f}$ on an open subset of $\mathbb{P}^1$.

e) Conclude that any rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ can be extended to a morphism $\mathbb{P}^1 \to \mathbb{P}^1$.

Exercise 2. (Cremona transformation, see Gathmann Exercise 9.29)
Let $a = (1 : 0 : 0), b = (0 : 1 : 0), c = (0 : 0 : 1)$ be the three coordinate points of $\mathbb{P}^2$ and $U = \mathbb{P}^2 \setminus \{a, b, c\}$. Consider the morphism

$$f : U \to \mathbb{P}^2, (x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1)\,.$$

a) Show that there is no morphism $\mathbb{P}^2 \to \mathbb{P}^2$ extending f.
Hint: For coordinates $(z_0 : z_1 : z_2)$ on the target, calculate $f^{-1}(V(z_i))$.

b) Show that f is dominant and $f \circ f \sim \mathrm{id}_{\mathbb{P}^2}$ as rational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Bonus exercise:

c) Let $\widetilde{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ at $\{a, b, c\}$. Show that f extends to a morphism $\widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$.
Note: In fact it even extends to a *isomorphism* $\widetilde{\mathbb{P}}^2 \to \widetilde{\mathbb{P}}^2$, called the Cremona transformation.