Presence Sheet 1

Exercise 1. The space $\mathrm{Mat}(m \times n) = \mathrm{Mat}(m \times n, K)$ of $m \times n$ matrices over the algebraically closed field K can be identified with the affine space $\mathbb{A}^{m \cdot n} = \mathbb{A}_K^{m \cdot n}$. Write $I_n \in \mathrm{Mat}(n \times n)$ for the identity matrix.

For the following subsets of various matrix spaces, decide whether they are Zariski open, closed, or neither.

a) $\mathrm{GL}(n) = \{A : A \text{ invertible}\} \subset \mathrm{Mat}(n \times n)$ general linear group

b) $\mathrm{SL}(n) = \{A : \det A = 1\} \subset \mathrm{Mat}(n \times n)$ special linear group

c) $\mathrm{Mat}^{\leq k}(m \times n) = \{A : \mathrm{rank} A \leq k\} \subset \mathrm{Mat}(m \times n)$ matrices of rank at most k

d) $\mathrm{Mat}^{\geq k}(m \times n) = \{A : \mathrm{rank} A \geq k\} \subset \mathrm{Mat}(m \times n)$ matrices of rank at least k

e) $U(n, \mathbb{C}) = \{A : A \cdot A^* = I_n\} \subset \mathrm{Mat}(n \times n, \mathbb{C})$ unitary matrices over $K = \mathbb{C}$

f) $\mathrm{Diag}(n) = \{A : A \text{ diagonal}\} \subset \mathrm{Mat}(n \times n)$ diagonal matrices

g) $\mathrm{Nil}(n) = \{A : A^m = 0 \text{ for some } m\} \subset \mathrm{Mat}(n \times n)$ nilpotent matrices

h) $\mathrm{Comm}(n) = \{(A, B) : A \cdot B = B \cdot A\} \subset \mathrm{Mat}(n \times n)^2$ pairs of commuting matrices

Solution. The determinant det is a polynomial function in the entries of the matrix. Hence $\mathrm{GL}(n) = \mathrm{Mat}(n \times n) \setminus V(\det)$ is Zariski open, and $\mathrm{SL}(n) = V(det - 1)$ is Zariski closed. Being of rank at most k is equivalent to the vanishing of all $(k+1) \times (k+1)$-minors of the matrix, which are again determinants of submatrices and hence polynomial. Thus $\mathrm{Mat}^{\geq k}(m \times n)$ is Zariski closed, and $\mathrm{Mat}^{\geq k}(m \times n) = \mathrm{Mat}(m \times n) \setminus \mathrm{Mat}^{\leq k-1}(m \times n)$ is Zariski open.

For $n = 1$ we have $U(1, \mathbb{C}) = S^1 \subset \mathbb{A}^1$ is the unit circle, which is infinite but not equal to $\mathbb{A}^1$ and thus not Zariski closed. Similarly its complement is not closed, so $U(1, \mathbb{C})$ is also not Zariski open.

The diagonal matrices are cut out by the vanishing of all non-diagonal entries of $A = (a_{i,j})_{i,j}$:

$$\mathrm{Diag}(n) = V(a_{i,j} : i \neq j)$$

Thus $\mathrm{Diag}(n)$ is Zariski closed.

From Linear Algebra, we know (using the Jordan decomposition) that an $n \times n$ matrix A is nilpotent if and only if $A^n = 0$. The entries of the matrix A^n are polynomials in the entries of A, and thus this is a Zariski closed condition (cut out by the vanishing of those polynomials).

Similarly, the entries of $A \cdot B - B \cdot A$ are polynomials in the entries of A, B, and thus $\mathrm{Comm}(n)$ is Zariski closed.

Exercise 2. Consider the set

$$\mathrm{Id}(2,\mathbb{C}) = \{A \in \mathrm{Mat}(2\times 2,\mathbb{C}) : A^2 = I_2\}$$

of idempotent matrices. Show that it can be written as the disjoint union of 3 non-empty affine varieties. (*Bonus*: If you have seen the definition of irreducible decomposition: compute it for $\mathrm{Id}(2,\mathbb{C})$.)
Hint: You can find the components by solving equations, or by using results from Linear Algebra concerning eigenvalues, etc ...

Solution. We have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If $a+d \neq 0$ this forces $b = c = 0$ and thus $a^2 = d^2 = 1$. This gives the solutions $a = d = 1$ and $a = d = -1$ (since $a = 1, d = -1$ and $a = -1, d = 1$ are excluded by $a + d \neq 0$).

If $a + d = 0$ the remaining equation is $a^2 + bc = 1$. Thus we have

$$\mathrm{Id}(2,\mathbb{C}) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \sqcup \left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \sqcup V(a+d, a^2+bc-1)\,.$$

All three sets are affine varieties, and they are also disjoint.

Bonus: The two one-point sets are clearly irreducible. For $X = V(a + d, a^2 + bc - 1)$ we compute

$$K[a,b,c,d]/\langle a+d, a^2+bc-1\rangle \cong K[a,b,c]/\langle a^2+bc-1\rangle\,.$$

We claim that the polynomial $f = a^2+bc-1$ is irreducible in the UFD $K[a, b, c]$. Indeed, we can see it as a polynomial of degree 1 in c. If we had a decomposition $f = h \cdot g$ then without loss of generality h, g have degrees $1, 0$ in c and we have

$$bc + a^2 - 1 = (h_1(a,b)c + h_2(a,b)) \cdot g(a,b)\,.$$

This forces $b = h_1(a, b) \cdot g(a, b)$, which up to units implies $h_1 = b, g = 1$ or $h_1 = 1, g = b$. In the first case we are done, in the second we get a contradiction since f is not divisible by b.

Alternatively, it is easy to see that the polynomial $bc-1$ is irreducible in $K[b, c]$, hence the polynomial $a^2 + bc - 1$ is irreducible in $K[a, b, c]$ by Eisenstein'c criterion.

Since the irreducible element f is prime, we have that $K[a, b, c, d]/\langle a + d, a^2 + bc - 1\rangle$ is a domain and hence $\langle a + d, a^2 + bc - 1\rangle$ is prime. By a result from the lecture, this shows that X is irreducible, so the decomposition of $\mathrm{Id}(2,\mathbb{C})$ above is the irreducible decomposition.