Presence Sheet 10

Exercise 1. (Basic schemes)

a) For the zero ring $R = \{0\}$ show that $\operatorname{Spec} R = \emptyset$.

b) For a field K, show that $\operatorname{Spec} K$ has a unique point. Are fields the only rings with this property?

c) For any ring R and ideal I, consider the quotient map $\varphi : R \to R/I$. Show that the map

$$\Phi : \operatorname{Spec} R/I \to \operatorname{Spec} R, q \mapsto \varphi^{-1}(q)$$

is injective with image $V(I)$. Moreover, show that the pullback of the Zariski topology is the Zariski topology.
Note: This means that φ induces a homeomorphism from $\operatorname{Spec} R/I$ to $V(I) \subseteq \operatorname{Spec} R$.

d) For K a field and $m \in \mathbb{N}_{>0}$, what is the spectrum $\operatorname{Spec} K[x]/\langle x^m \rangle$ as a topological space?

Bonus exercise:

e) What is the spectrum $\operatorname{Spec} K[[t]]$ of the formal power series ring $K[[t]]$ in a single variable over a field K, as a topological space?

Solution.

a) The only ideal of $R = \{0\}$ is R itself, which is not prime (since prime ideals must be proper ideals). Thus there are no prime ideals and hence the spectrum is empty.

b) Any nonzero element of K is a unit, and thus cannot be contained in any prime ideal. Thus $\langle 0 \rangle$ is the unique prime ideal of K and element of $\operatorname{Spec} K$. In part d) below we see a counter-example of a non-reduced ring with single-point spectrum.

c) The map is well-defined since preimages of prime ideals under ring homomorphisms are prime ideals. Since $\{0\} \subseteq q$, we also have $I = \varphi^{-1}(\{0\}) \subseteq \varphi^{-1}(q)$. This shows that Φ has image in $V(I) = \{p : I \subseteq p\} \subseteq \operatorname{Spec} R$. We claim that the map

$$\Psi : V(I) \to \operatorname{Spec} R/I, p \mapsto \varphi(p) = \{\varphi(a) : a \in p\} \subseteq R/I$$

is the inverse of Φ. To see this:

- $\Psi(\Phi(q)) = \varphi(\varphi^{-1}(q)) = q$ since φ is surjective.

- $\Phi(\Psi(p)) = \varphi^{-1}(\varphi(p))$; this certainly contains p. On the other hand, for $b \in \varphi^{-1}(\varphi(p))$ there exists $a \in p$ such that $\varphi(b) = \varphi(a) \in R/I$, so there exists $c \in I$ with $b = a + c$. But $a \in p$ and $c \in I \subseteq p$ showing $b \in p$ and thus the other inclusion.

Moreover, we have $\Phi^{-1}(V(J)) = V((J+I)/I)$ is closed, and conversely any closed subset $V(\widetilde{J})$ of $\operatorname{Spec} R/I$ (coming from some $\widetilde{J} \subset R/I$) is the preimage of $V(\varphi^{-1}(\widetilde{J}))$.

d) By the previous exercise it is given by

$$V(x^m) = \{p \in \operatorname{Spec} K[x] : x^m \in p\} = \{p \in \operatorname{Spec} K[x] : x \in p\} = \{\langle x \rangle\},$$

where in the second equality we used that x^m is contained in the prime ideal p iff x is contained in p, by the defining property of prime ideals. Thus this spectrum is a single-point space.

e) The ring $R = K[[t]]$ is a discrete valuation ring, and thus $\langle 0 \rangle$ and $\langle t \rangle$ are the only prime ideals. Apart from the total space $\operatorname{Spec} R$ and the empty set $\emptyset$, the only closed set is $\{\langle t \rangle\} = V(t)$. If there was an ideal $J \subseteq R$ with $V(J) = \{\langle 0 \rangle\}$ it would have to satisfy $J \subseteq \langle 0 \rangle$ and thus $J = \langle 0 \rangle$, giving a contradiction since $V(\langle 0 \rangle) = \operatorname{Spec} R$ is the full space. Thus there are precisely the above three closed sets in the two-point space $\operatorname{Spec} R$.

Exercise 2. (Zariski topology) Let R be a ring.

a) Show that for $p \subseteq R$ a prime ideal, the vanishing set $V(p)$ is irreducible.
Hint: There is a one-line argument using [Gathmann, Remark 12.9 (b)].

b) Let S be a reduced ring with $\operatorname{Spec}(S)$ irreducible. Show that S is an integral domain.
Hint: Remember that in any ring, the nilradical $\sqrt{\langle 0 \rangle}$ is given by the intersection of all prime ideals of the ring.

c) Show that the irreducible closed subsets of $\operatorname{Spec} R$ are exactly given by $V(p)$ for $p \subseteq R$ a prime ideal.
Hint: For $V(J) \subseteq \operatorname{Spec}(R)$ closed, show that the map $\Phi : \operatorname{Spec}(R/\sqrt{J}) \to V(J)$ from Exercise 1 is a homeomorphism.

d) Conclude that $\dim \operatorname{Spec} R$ is given by the Krull dimension of R.

Solution.

a) We saw that $V(p) = \overline{\{p\}}$ is the closure of a one-point set, and thus irreducible as the closure of an irreducible space.

b) Assume $f \cdot g = 0 \in S$, then $\operatorname{Spec}(S) = V(0) = V(fg) = V(f) \cup V(g)$. Since $\operatorname{Spec}(S)$ is irreducible, we have that e.g. $V(f) = \operatorname{Spec}(S)$, meaning that $f \in p$ for all $p \in \operatorname{Spec}(S)$. Thus f is contained in the intersection of all prime ideals of S, which is the nilradical. Since S is reduced, the nilradical is given by $\langle 0 \rangle$ so $f = 0$.

c) One direction was shown in part a). For the other assume that $Y = V(J) \subseteq \operatorname{Spec}(R)$ is an irreducible closed set. For $I = \sqrt{J}$ we know that $V(I) = V(J)$ and by Exercise 1 c) we have $Y \cong \operatorname{Spec}(R/I)$ as topological spaces. But $S = R/I$ is reduced and $\operatorname{Spec}(R/I) \cong Y$ is irreducible, hence R/I is a domain. This shows I is prime, and so $Y = V(I)$ is of the desired form.

d) By part c) and the scheme-theoretic Nullstellensatz, there is an inclusion-reversing bijection between prime ideals of R and irreducible closed subsets of $\mathrm{Spec}(R)$. The dimension and Krull dimension are defined as the supremum of lengths of chains of these, and thus agree.