Presence Sheet 11

Exercise 1. (Tangent spaces) For two schemes X and Y over a base scheme S define the set of Y-points of X to be $\mathrm{Mor}_S(Y, X)$ and denote it by $X(Y)$. For a point x of a locally ringed space X the cotangent space of X at x is the $K(x)$–vector space

$$T^*_{X,x} := \mathrm{m}(x)/\mathrm{m}(x)^2\,,$$

where $\mathrm{m}(x) \subseteq \mathcal{O}_{X,x}$ is the maximal ideal. The tangent space of X at x is the $K(x)$-dual of $T^*_{X,x}$ and is denoted by $T_{X,x}$.

Suppose X is a variety over an algebraically closed field K. Consider schemes over $\operatorname{Spec} K$.

a) Show that $X(\operatorname{Spec} K)$ is naturally identified with the set of closed points of X. Is this true if K is not necessarily algebraically closed?

b) Show that any element of $X(\operatorname{Spec} K[t]/t^2)$ is naturally identified with a choice of a closed point $x \in X$ and a choice of an element $v \in T_{X,x}$.

c) The homomorphisms $K \hookrightarrow K[t]/t^2 \twoheadrightarrow K$ (where $t \mapsto 0$ in the second homomorphism) correspond to morphisms of schemes $\operatorname{Spec} K \to \operatorname{Spec} K[t]/t^2 \to \operatorname{Spec} K$. Describe the induced maps of sets $X(\operatorname{Spec} K) \to X(\operatorname{Spec} K[t]/t^2) \to X(\operatorname{Spec} K)$ in terms of the data above.

d) Let $f\colon X \to Y$ be a morphism of varieties over K and $x \in X$ be a closed point. Construct a natural linear morphism $df_x\colon T_{X,x} \to T_{Y,f(x)}$. You can use the definition of tangent space or the description from b).

Let Z be the fibre of f above $f(x)$. Show that $\mathrm{Ker}(df_x) \simeq T_{Z,x}$.

e) Consider $X = \mathbb{A}^n_K = \operatorname{Spec} K[x_1, \cdots x_n]$ and $o = (0, \cdots, 0) \in X$. Show that the cotangent space $T^*_{X,o}$ is identified with the standard n-dimensional space K^n via $\mathrm{m}(o) \ni x_i \mapsto e_i$. By taking the dual basis we get a similar identification of $T_{X,o}$ with K^n. Now let $a = (a_1, \cdots, a_n)$ be an arbitrary closed point of $\mathbb{A}^n_K$. By the translation morphism $p \mapsto p + a$ of $\mathbb{A}^n_K$ identify $T_{X,a}$ with $T_{X,o} \simeq K^n$ using d).

f) Let $X := \mathbb{A}^{n+1}_K \backslash \{(0, 0, \cdots, 0)\} \to \mathbb{P}^n_K$ be the natural morphism. Recall that geometrically it maps a closed point $a = (a_0, \cdots, a_n)$ to the line ℓ_a through o and a. Using e) identify any tangent space at any closed point of $X \subset \mathbb{A}^{n+1}_K$ with the standard K^{n+1}. Show that $T_{X,a} \to T_{\mathbb{P}^n_K, \ell_a}$ is surjective and describe $T_{\mathbb{P}^n_K, \ell_a}$ as a quotient of K^{n+1}.

Solution.

a) A morphism $\operatorname{Spec} K \to X$ defines a point by taking the morphism of underlying topological spaces. The residue field of ths point is K so the point is closed. On the other hand any closed point has a residue field isomorphic to K and thus defines a local homomorphism $\mathcal{O}_{X,x} \twoheadrightarrow K$ which is the same as a morphism $\operatorname{Spec} K \to X$

b) Given a morphism $\operatorname{Spec} K[t]/t^2 \to X$ taking the composition with the closed embedding $\operatorname{Spec} K \hookrightarrow \operatorname{Spec} K[t]/t^2$ we obtain a morphism $\operatorname{Spec} K \to X$ i.e. a closed point $x \in X$. Now to give a morphism of schemes $\operatorname{Spec} K[t]/t^2 \to X$ giving rise to the closed point x is equivalent to giving a local homomorphism $\phi\colon \mathcal{O}_{X,x} \to K[t]/t^2$. As $\phi(\mathrm{m}_x) \subset tK$ we have $\phi(\mathrm{m}_x^2) = 0$ so any such ϕ defines a linear map $\tilde{\phi}\colon T^*_{X,x} = \mathrm{m}_x/\mathrm{m}_x^2 \to tK \simeq K$ i.e. an element in $T_{X,x}$. On the other hand, given such a linear map $\tilde{\phi}$ we can define a local homomorphism $\phi\colon \mathcal{O}_{X,x} \to K[t]/t^2$ as follows: use morphisms $K \to \mathcal{O}_{X,x}/\mathrm{m}_x^2 \to K$ to split $\mathcal{O}_{X,x}$ as $K \oplus \mathrm{m}_x/\mathrm{m}_x^2$ and define $\phi(x,m) = x + t\tilde{\phi}(\overline{m})$.

c) As we can see from b) the closed embedding $\operatorname{Spec} K \hookrightarrow \operatorname{Spec} K[t]/t^2$ induces the map $X(\operatorname{Spec} K[t]/t^2) \to X(\operatorname{Spec} K)$ which maps (x,v) to x. The map $\operatorname{Spec} K[t]/t^2 \to \operatorname{Spec} K$ corresponds to the structure morphism $K \hookrightarrow K[t]/t^2$ and any local homomorphism $\mathcal{O}_{X,x}$ defined by composing a local homomorphism $\mathcal{O}_{X,x} \to K$ with the structure morphism maps m_x to 0 and hence defines a map $X(\operatorname{Spec} K) \to X(\operatorname{Spec} K[t]/t^2)$ sending x to $(x,0)$.

d) As f is a morphism of schemes it defines a local homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ therefore inducing a linear map $\mathrm{m}_{f(x)}/\mathrm{m}_{f(x)}^2 \to \mathrm{m}_x/\mathrm{m}_x^2$. Taking the dual we obtain the required morphism. An equivalent way to construct df_x is to consider an element $v \in T_{X,x}$ and to compose the associated morphism $\operatorname{Spec} K[t]/t^2$ with f to obtain a morphism $\operatorname{Spec} K[t]/t^2 \to Y$. The induced morphism $\operatorname{Spec} K \to Y$ then defines the point $f(x)$ hence the composition defines an element $df_x(v) \in T_{Y,f(x)}$. It's easy to see that $d(f \circ g) = df \circ dg$

Using the second description from a morphism $v\colon \operatorname{Spec} K[t]/t^2 \to X$ defines an element in $\operatorname{Ker}(df_x)$ iff its composition with f factors through $\operatorname{Spec} K$ given by the embedding of $f(x)$. By the universal propertyt of the fibre product this is equivalent to the fact that v factors through $Z \hookrightarrow X$ and hence defines a vector in $T_{Z,x}$

e) The maximal ideal $\mathrm{m}(o) \trianglelefteq K[x_1, \cdots, x_n]$ of o is generated by $x_1, \cdots, x_n$ and $\mathrm{m}(o)/\mathrm{m}(o)^2$ is freely generated by $\overline{x_1}, \cdots, \overline{x_n}$ as a K-vector space hence the identification follows. The translation morphism $t \mapsto t + a$ is an isomorphism from $\mathbb{A}^n_K \to \mathbb{A}^n_K$ with the inverse given by $t \mapsto t - a$ so we obtain an isomorphism $T_{X,0} \simeq T_{X,a}$ and hence the identification.

f) Let us call this morphism π. The preimage of $\mathbb{A}^n_K \simeq \{a_0 \neq 0\} =: U \subset \mathbb{P}^n_K$ is given by $\{a_0 \neq 0\} \simeq \mathbb{G}_m \times \mathbb{A}^n_K =: V \subset X$. Without loss of generality assume $\ell_a \in U$. The restriction of π to V is given by $(t, x_1, \cdots, x_n) \mapsto (\frac{x_1}{t}, \cdots \frac{x_n}{t})$. In particular $\pi|_V$ has a section given by $(y_1, \cdots, y_n) \mapsto (1, y_1, \cdots y_n)$. Hence the associated map $T_{\mathbb{A}^{n+1}_K, a} \to T_{\mathbb{P}^n_K, \ell_a}$ also has a section so it is surjective.

Now considering the affine charts U, V we see that the fibre of π over ℓ_a is given by $\mathbb{G}_m \hookrightarrow X$ via $t \mapsto (a_0 t, \cdots, a_n t)$. This morphism maps a generator $1 \in K \simeq T_{\mathbb{G}_m, 1}$ to the vector $(a_0, \cdots, a_n) \in K^{n+1} \simeq T_{X,a}$ i.e. to a generator of ℓ_a. Hence using e) we get $T_{\mathbb{P}^n_K, \ell_a} \simeq K^{n+1}/\ell_a$.