Presence Sheet 12

Exercise 1. (Subschemes) Are the following morphisms examples of open subschemes, closed subschemes or neither?

a) $f : X = \operatorname{Spec} K[x,y]/(xy-1) \to \operatorname{Spec} K[x] = Y$ unique map with

$$f^* : K[x] \to K[x,y]/(xy-1), x \mapsto x$$

b) $f : \mathbb{P}^1 \to \mathbb{P}^2, (x_0 : x_1) \mapsto (x_0 : x_1 : 0)$

c) $f : \mathbb{A}^1 \to \mathbb{A}^2, t \mapsto (t^2, t^3)$

Solution.

a) We have $K[x,y]/(xy-1) \cong K[x]_x$ with $y \mapsto 1/x$. Thus $X = D(x) \subseteq Y$ is a distinguished affine open subset, making the morphism f an open embedding (or subscheme).

b) The target scheme $\mathbb{P}^2$ has a cover $\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$ by three copies of $\mathbb{A}^2$. Restricting f to the preimage of U_0, U_1 we obtain morphisms of the form

$$f_i : \operatorname{Spec} K[x] = \mathbb{A}^1 \to \mathbb{A}^2 = \operatorname{Spec} K[x,y] \text{ from } f_i^* : K[x,y] \to K[x], x \mapsto x, y \mapsto 0\,.$$

Since f_i^* are surjective, we see that $f|_{f^{-1}(U_i)}$ is a closed subscheme of U_i for $i = 0, 1$. For $i = 2$ we have $f^{-1}(U_2) = \emptyset$, which is also a closed subscheme of U_2, corresponding to the surjective map $K[x,y] \to \{0\}$ as $\emptyset = \operatorname{Spec}\{0\}$ is the spectrum of the zero-ring.

c) The map f is not an open embedding since its image $V(x^3 - y^2)$ is not open in $\mathbb{A}^2$. Otherwise $\mathbb{A}^2 = V(x^3 - y^2) \cup (\mathbb{A}^2 \setminus V(x^3 - y^2))$ would be a decomposition into two closed and disjoint sets (since $V(x^3 - y^2)$ is certainly also closed), making $\mathbb{A}^2$ disconnected, which we know is not the case.

However, f is also not a closed subscheme, since the induced map of rings

$$K[x,y] \to K[t], x \mapsto t^2, y \mapsto t^3$$

is not surjective (as t is not in the image).

Exercise 2. (Fiber products)

a) Let X, S be schemes, then the set of *S-points of X* is given by

$$X(S) = \{S \xrightarrow{f} X : f \text{ morphism}\}\,.$$

For $g : X \to Y$ a morphism of schemes, there is a natural map

$$X(S) \to Y(S), f \mapsto g \circ f$$

of their S-points.

Show that for $f_X : X \to Z$ and $f_Y : Y \to Z$ two morphisms of schemes, we have

$$(X \times_Z Y)(S) = \{(x, y) \in X(S) \times Y(S) : f_X \circ x = f_Y \circ y \in Z(S)\}\,.$$

Note: This makes precise the idea that points of $X \times_Z Y$ are pairs of points of X, Y mapping to the same point in Z.

b) Calculate the following fiber products $X \times_Z Y$:

i) $X = \mathbb{A}^1 \xrightarrow{x \mapsto (x,0)} \mathbb{A}^2 = Z$ and $X = \mathbb{A}^1 \xrightarrow{y \mapsto (0,y)} \mathbb{A}^2 = Z$

ii) $X = \mathbb{A}^1 \xrightarrow{t \mapsto (t,t)} \mathbb{A}^2 = Z$ and $X = \mathbb{A}^3 \xrightarrow{(x,y,z) \mapsto (x,y)} \mathbb{A}^2 = Z$

Hint: Remember that for $\varphi : S \to R$ a ring homomorphism and $I \subseteq S$ an ideal we have $R \otimes_S (S/I) \cong R/\langle \varphi(I) \rangle$.

c) Show that for a fiber product

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_Y} & Y \\ \big\downarrow{\scriptstyle \pi_X} & & \big\downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{f_X} & Z \end{array}$$

such that f_X is a closed subscheme, also π_Y is a closed subscheme.

d) Is it true that for X, Y, Z varieties also $X \times_Z Y$ is a variety?

Solution.

a) This is just the universal property of the fiber product: a morphism $S \to X \times_Z Y$ is the same thing as a pair of morphisms $S \to X$ and $S \to Y$ giving the same map to Z when composed with f_X, f_Y.

b) All schemes are affine, so their fiber product is the spectrum of the tensor product of the associated rings.

i) We have

$$K[x] \otimes_{K[x,y]} K[y] = K[x] \otimes_{K[x,y]} K[x,y]/\langle x \rangle \cong K[x]/\langle x \rangle \cong K\,.$$

Thus $X \times_Z Y = \operatorname{Spec} K$ mapping to X, Y as the inclusion of the origin.

ii) We have $K[t] \cong K[x,y]/\langle x-y\rangle$ via $x \mapsto t, y \mapsto t$ and using this, we see

$$K[t] \otimes_{K[x,y]} K[x,y,z] \cong K[x,y]/\langle x-y\rangle \otimes_{K[x,y]} K[x,y,z] \cong K[x,y,z]/\langle x-y\rangle \cong K[x,z]\,.$$

Thus $X \times_Z Y \cong \mathbb{A}^2$ mapping to X, Y via

$$\mathbb{A}^2 \xrightarrow{(x,z)\mapsto x} X = \mathbb{A}^1 \text{ and } \mathbb{A}^2 \xrightarrow{(x,z)\mapsto(x,x,z)} Y = \mathbb{A}^3\,.$$

c) We prove that π_Y is a closed subscheme by covering it with affine schemes and calculating the restriction of π_Y over these. To obtain this cover, we *first* cover Z with affine open schemes $U_i = \operatorname{Spec}(R_i)$. Since f_X is a closed subscheme, the restriction of f_X over U_i is given by

$$\operatorname{Spec} R_i/I_i \to \operatorname{Spec} R_i = U_i$$

for some ideal $I_i \subseteq R_i$. Now take the open subscheme $f_Y^{-1}(U_i) \subseteq Y$ and cover it by affine subschemes $V_{ij} = \operatorname{Spec} S_{ij}$ mapping to $U_i = \operatorname{Spec}(R_i)$ with associated ring homomorphism $f_{ij} : R_i \to S_{ij}$. Then we have seen that $\pi_Y^{-1}(V_{ij})$ is given by

$$\operatorname{Spec}(R_i/I_i) \otimes_{R_i} S_{ij} \cong \operatorname{Spec} S_{ij}/\langle f_{ij}(I_i)\rangle\,.$$

Since the associated map $S_{ij} \to S_{ij}/\langle f_{ij}(I_i)\rangle$ is surjective, we have that π_Y is a closed subscheme as desired.

d) No: take $X = V(y) \subseteq \mathbb{A}^2 = Z$ and $Y = V(y - x^2) \subseteq \mathbb{A}^2 = Z$. All three schemes come from affine varieties, but their fiber product (which is just the scheme-theoretic intersection) is

$$V(y, y-x^2) = V(y, x^2) \cong \operatorname{Spec} K[x,y]/\langle y, x^2\rangle \cong K[x]/\langle x^2\rangle\,.$$

This is not reduced and thus not a variety.