Presence Sheet 13

In the sheet below we discuss sheaves on topological spaces X which are not schemes (and sometimes sheaves which are not $\mathcal{O}_X$-modules). However, all definitions relevant for the exercises below (stalks, sheafification, injective and surjective morphisms of sheaves, etc) still make sense, even though we presented them for schemes.

Exercise 1. (Sheaffification of the constant presheaf) Let X be a topological space and S a set. Let $\mathcal{F}$ be the presheaf (of sets) given by

$$\mathcal{F}(U) = \{f : U \to S : f \text{ constant}\} \text{ for } U \subseteq X \text{ open}.$$

a) Show that the stalk $\mathcal{F}_p$ at any $p \in X$ is isomorphic to S.

b) Define the sheaf $\underline{S}$ of locally constant functions to S.
Hint: If you're unsure what "locally constant" means, ask wikipedia, ChatGPT or the lecturer.

c) Show that $\underline{S}$ is the sheafification of $\mathcal{F}$.
Hint: You can use the definition directly or prove that $\mathcal{G}$ is a sheaf and that the natural map $\mathcal{F} \to \underline{S}$ is an isomorphism on stalks.

Solution.

a) To show that the stalk $\mathcal{F}_p$ at any $p \in X$ is isomorphic to S, we first recall the definition of the stalk of a presheaf. The stalk $\mathcal{F}_p$ is given by:

$$\mathcal{F}_p = \bigcup_{p \in U} \mathcal{F}(U) / \sim,$$

where $(U, \varphi) \sim (V, \varphi|_V)$ for $p \in V \subseteq U$.

Since $\mathcal{F}(U)$ consists of constant functions from U to S, any element of $\mathcal{F}(U)$ is uniquely determined by its value at any point in U (e.g. at p itself). Thus, the germ of a section at p is determined by its value at p. Therefore, each germ at p corresponds to an element of S, giving us:

$$\mathcal{F}_p \cong S.$$

b) The sheaf $\underline{S}$ of locally constant functions to S is defined as follows. For each open set $U \subseteq X$, let

$$\underline{S}(U) = \{f : U \to S \mid f \text{ is locally constant}\}.$$

A function $f : U \to S$ is called locally constant if for each point $x \in U$, there exists an open neighborhood $V \subseteq U$ of x such that $f|_V$ is constant.

c) To show that $\underline{S}$ is the sheafification of $\mathcal{F}$, we use the definition of sheafification. According to the sheafification process, a section of the sheafification on an open set U is given by a collection $(s_p)_{p\in U}$ of elements of the stalks $\mathcal{F}_p$, which are locally around each point of U given by sections of the presheaf $\mathcal{F}$.

Let's break down the steps:

(a) For each point $p \in U$, we have $s_p \in \mathcal{F}_p$. Since $\mathcal{F}_p \cong S$, we can view s_p as an element of S.

(b) Locally around each point $p \in U$, there exists an open neighborhood $V \subseteq U$ of p and a section $f_V \in \mathcal{F}(V)$ such that the germ of f_V at p corresponds to s_p.

(c) Since $\mathcal{F}(V)$ consists of constant functions, f_V is a constant function on V taking the value $s_p \in S$.

(d) The collection $(s_p)_{p\in U}$ corresponds to a locally constant function $f : U \to S$, where $f(p) = s_p$ for each $p \in U$. This is precisely a section of $\underline{S}$ over U.

Therefore, $\underline{S}$ is the sheafification of $\mathcal{F}$, as a section of $\underline{S}$ on U is a collection $(s_p)_{p\in U}$ of elements of the stalks, locally around each point of U given by a section of $\mathcal{F}$.

Exercise 2. (Exponential exact sequence reloaded) Recall from Presence sheet 3 that for the complex numbers $X = \mathbb{C}$ with the Euclidean topology and $U \subseteq \mathbb{C}$ open, we have two sheaves

$$\mathcal{O}^{\text{hol}}(U) = \{f : U \to \mathbb{C} : f \text{ holomorphic}\},$$
$$\mathcal{O}^{\text{hol},\times}(U) = \{f : U \to \mathbb{C}^* : f \text{ holomorphic}\}.$$

on X and the exponential map $\exp : \mathcal{O}^{\text{hol}} \to \mathcal{O}^{\text{hol},\times}$ given by

$$\exp_U : \mathcal{O}^{\text{hol}}(U) \to \mathcal{O}^{\text{hol},\times}(U), f \mapsto \exp(f).$$

a) Show that exp is a surjective map of sheaves (of abelian groups).
Bonus question: Is $\exp_U$ surjective for all $U \subseteq X$ open?

b) What is the kernel sheaf of exp?
Hint: Exercise 1.

c) With the two previous exercise parts in mind, what do you think is the exponential exact sequence?

Solution.

a) To show that exp is a surjective map of sheaves, we check surjectivity at the level of stalks.

Let $p \in U$ and consider the stalks $\mathcal{O}_p^{\text{hol}}$ and $\mathcal{O}_p^{\text{hol},\times}$. The map induced on stalks by exp is:

$$\exp_p : \mathcal{O}_p^{\text{hol}} \to \mathcal{O}_p^{\text{hol},\times}.$$

A germ of a section at p is represented by (V, ϕ), where V is an open neighborhood of p and ϕ is a holomorphic function on V, and via $\exp_p$ it's sent to $[(V, \exp(\phi))]$.

For any germ $(V, \psi) \in \mathcal{O}_p^{\text{hol},\times}$, where ψ is a non-vanishing holomorphic function on V, we can locally find a logarithm of ψ, i.e., a holomorphic function ϕ on some possibly smaller neighborhood $W \subseteq V$ such that $\exp(\phi) = \psi$ on W. This follows since a logarithm-function exists on a small disc D around the point $\psi(p) \in \mathbb{C} \setminus \{0\}$. Then we can take $W = \psi^{-1}(D)$ and $\phi = \log(\psi)$. Since $\exp_p[(W, \phi)] = [(W, \psi|_W)] = [(V, \psi)] \in \mathcal{O}_p^{\mathrm{hol},\times}$ we conclude that $\exp_p$ is surjective.

For the bonus question, consider $U = \mathbb{C}^*$. The function $f(z) = z$ is a holomorphic section of $\mathcal{O}^{\mathrm{hol},\times}(U)$. Suppose there exists $g \in \mathcal{O}^{\mathrm{hol}}(U)$ such that $\exp(g) = z$. Then g would be a single-valued holomorphic logarithm defined on all of $\mathbb{C}^*$. This is impossible (do you remember why? It has to do with the fact that $\int_{\partial B_1(0)} \frac{1}{z} = 2\pi i \neq 0$).

b) The kernel sheaf of exp, denoted by ker(exp), consists of all holomorphic functions f such that $\exp(f) = 1$. This means f must take values in the set of complex numbers whose exponential is 1, which is the set $S = 2\pi i\mathbb{Z}$. If a function $U \to S \subseteq \mathbb{C}$ is holomorphic (i.e. in $\mathcal{O}^{\mathrm{hol}}(U)$) then it's certainly also continuous. Since S carries the discrete topology, this is equivalent to the function being locally constant. Conversely any locally constant function is of course holomorphic. Thus we have

$$\ker(\exp) = \underline{2\pi i\mathbb{Z}}.$$

This is the constant sheaf on $X = \mathbb{C}$ with values in $2\pi i\mathbb{Z}$.

c) The exponential exact sequence is the exact sequence of sheaves

$$0 \to \underline{2\pi i\mathbb{Z}} \to \mathcal{O}^{\mathrm{hol}} \xrightarrow{\exp} \mathcal{O}^{\mathrm{hol},\times} \to 0\,.$$