Presence Sheet 2

Exercise 1. Show that a map $F : \mathbb{A}^n \to \mathbb{A}^m$ of the form $F(x) = (f_1(x), \ldots, f_m(x))$ with $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ is continuous (with respect to the Zariski topology).
Note: Such polynomial maps will be examples of morphisms of affine varieties later, but for the purpose of the exercises below we just need their continuity.

Solution. For a closed set $Y = V(g_1(y), \ldots, g_r(y)) \subseteq \mathbb{A}^m$ we have

$$F^{-1}(Y) = V(g_i(f_1(x), \ldots, f_m(x)) : i = 1, \ldots, r) \subseteq \mathbb{A}^n$$

is a again a vanishing locus of polynomials, hence closed.

Exercise 2. Given $d \in \mathbb{N}$, we can identify the set $P_d \subseteq K[x]$ of monic degree d polynomials in with $\mathbb{A}^d$ by the map

$$\mathbb{A}^d \xrightarrow{\sim} P_d, (a_0, \ldots, a_{n-1}) \mapsto x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0 .$$

a) Show that the set $\Delta_2 \subseteq P_2$ of degree 2 polynomials with a double zero is a Zariski closed subset.

b) Show that the set $\Delta_d \subseteq P_d$ of degree d polynomials with a double zero is a hypersurface.
Hint: If you are stuck, you can try to google the word "discriminant".

c) Write down a polynomial map $F : \mathbb{A}^{d-1} \to P_d$ whose image is Δ_d, and conclude that Δ_d is irreducible.

Solution.

a) A polynomial $f = x^2 + a_1 x + a_0$ has a double zero iff the discriminant $a_1^2 - 4a_0$ vanishes, which cuts out a closed subset of P_2.

b) Again, $f \in P_d$ has a double zero iff its discriminant vanishes, which is a (nonzero) polynomial of degree $\binom{d}{2}$ in the coefficients $a_{d-1}, \ldots, a_0$. Thus its vanishing locus Δ_d is a hypersurface as claimed.

c) The desired polynomial map is

$$F(a_1, \ldots, a_{d-1}) = (x - a_1)^2 (x - a_2) \cdots (x - a_{d-1}) \in \Delta_d \subseteq P_d .$$

Expanding the products out shows that the coefficients of $F(a_1, \ldots, a_{d-1})$ depend polynomially on $a_1, \ldots, a_{d-1}$. Thus F is continuous by Exercise 1 and since its domain $\mathbb{A}^{d-1}$ is irreducible, its image $\Delta_d = F(\mathbb{A}^{d-1})$ is likewise irreducible (as shown on Exercise sheet 2).

Exercise 3. (Cayley-Hamilton theorem)
In this exercise we show the Cayley-Hamilton theorem from linear algebra. It says that for $A \in \mathrm{Mat}(n \times n, K)$ with characteristic polynomial $\chi_A(x) = \det(xE_n - A)$ we have $\chi_A(A) = 0$.

a) Show (or convince yourself) that the maps

$$\begin{aligned} \chi : &\quad \mathrm{Mat}(n \times n, K) \to P_n, A \mapsto \chi_A \\ \mathrm{ev} : &\quad \mathrm{Mat}(n \times n, K) \times P_m \to \mathrm{Mat}(n \times n, K), (A, f) \mapsto f(A) \end{aligned}$$

are polynomial maps in the sense of Exercise 1.

b) Show that the set $U = \{A \in \mathrm{Mat}(n \times n, K) : A \text{ has } n \text{ distinct eigenvalues}\}$ is a non-empty irreducible Zariski open subset of $\mathrm{Mat}(n \times n, K)$.

c) Prove the Cayley-Hamilton theorem for $A \in U$.
Hint: Note that for $S \in \mathrm{GL}(n, K)$ and $A \in \mathrm{Mat}(n \times n, K)$ we have $f(SAS^{-1}) = Sf(A)S^{-1}$ for any $f \in K[x]$.

d) Conclude that the Cayley-Hamilton theorem holds for all $A \in \mathrm{Mat}(n \times n, K)$.

Solution.

a) The formulas for the determinant in $\chi_A(x) = \det(xE_n - A)$ and the matrix powers and additions in $f(A)$ show that these expressions depend polynomially on the entries of A (and coefficients of f).

b) We have $U = \chi^{-1}(P_n \setminus \Delta_n)$ is the preimage of the Zariski open set $P_n \setminus \Delta_n$ and thus open. Moreover, given n distinct elements $\lambda_1, \ldots, \lambda_n$ the diagonal matrix $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is contained in U, so U is non-empty (here we use that algebraically closed fields are infinite).

c) Any matrix $A \in U$ is diagonalizable, so there is $S \in \mathrm{GL}(n, K)$ with $SAS^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) =: D$. Then

$$\chi_A(A) = S\chi_A(D)S^{-1} = S\mathrm{diag}(\chi_A(\lambda_1), \ldots, \chi_A(\lambda_n))S^{-1} = 0$$

since the eigenvalues λ_i are zeros of χ_A by definition.

d) Consider the composition

$$F : \mathrm{Mat}(n \times n, K) \xrightarrow{(\mathrm{id}, \chi)} \mathrm{Mat}(n \times n, K) \times P_n \xrightarrow{\mathrm{ev}} \mathrm{Mat}(n \times n, K)$$

It is continuous as the composition of polynomial maps, so $F^{-1}(\{0\}) \subseteq \mathrm{Mat}(n \times n, K)$ is closed. By part c) this closed set contains the non-empty open U, and since $\mathrm{Mat}(n \times n, K)$ is irreducible. Thus U is dense and so F is constant equal to 0, proving Cayley-Hamilton.

Exercise 4. Compute the dimension of the sets

$$T = \{A \in \mathrm{Mat}(2 \times 2, K) : \mathrm{trace}(A) = 0\}, \mathrm{Nil}_2 = \{A \in \mathrm{Mat}(2 \times 2, K) : A \text{ nilpotent}\}.$$

Solution. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $A(T) = K[a, b, c, d]/(a - d) \cong K[a, b, c]$ is a domain of Krull dimension 3, so T is irreducible of dimension 3. The subset $\mathrm{Nil}_2 \subseteq T$ is cut out by the vanishing of the determinant $-a^2 - bc$, which is a nonzero element of $A(T)$. Thus by Krull's principal ideal theorem, the set Nil_2 is of pure dimension $3 - 1 = 2$.