Presence Sheet 3

Note: Since the Presence Sheet 2 was pretty long, the beginning of the class will discuss the remaining exercises there. Below we just have some short exercises on the material from week 3.

Exercise 1. For the complex numbers $X = \mathbb{C}$ with the Euclidean topology and $U \subseteq \mathbb{C}$ open, let

$$\mathcal{O}^{\text{hol}}(U) = \{f : U \to \mathbb{C} : f \text{ holomorphic}\},$$
$$\mathcal{O}^{\text{hol},\times}(U) = \{f : U \to \mathbb{C}^* : f \text{ holomorphic}\}\,.$$

a) Convince yourself that $\mathcal{O}^{\text{hol}}$ and $\mathcal{O}^{\text{hol},\times}$ are sheaves of abelian groups (with respect to pointwise addition and multiplication).

b) Show that the maps

$$\exp_U : \mathcal{O}^{\text{hol}}(U) \to \mathcal{O}^{\text{hol},\times}(U), f \mapsto \exp(f)$$

are well-defined group homomorphisms that are compatible under restriction maps. *Note:* Such a collection of maps gives rise to a *morphism of sheaves*

$$\exp : \mathcal{O}^{\text{hol}} \to \mathcal{O}^{\text{hol},\times}\,.$$

Show that exp induces a well-define map $\exp_0 : \mathcal{O}^{\text{hol}}_0 \to \mathcal{O}^{\text{hol},\times}_0$ of the stalks at $0 \in \mathbb{C}$.

c) Show the the map

$$T : \mathcal{O}^{\text{hol}}_0 \to \mathbb{C}[[z]], f \mapsto \sum_{k\geq 0} \frac{f^{(k)}(0)}{k!}(0) \cdot z^k$$

to the formal power series ring $\mathbb{C}[[z]]$ is well-defined and injective. What is its image?

Bonus questions for (complex) analysis enthusiasts:

d) Are the group homomorphisms $\exp_U$ injective (resp. surjective) for the unit disc $U = B_1(0)$ (resp. $U \subseteq \mathbb{C}$ any open subset)?

e) Compute the kernel and the image of the map $\exp_0$ between the stalks at 0. Use this to calculate the stalk $\mathcal{O}^{\text{hol},\times}_0$.

f) On $X = \mathbb{R}$ with the Euclidean topology consider the sheaf $\mathcal{C}^\infty$ of infinitely differentiable (or smooth) functions to $\mathbb{R}$. Is the analogous Taylor expansion map $T : \mathcal{C}^\infty_0 \to \mathbb{R}[[x]]$ still injective?

Solution.

a) Given $f, g \in \mathcal{O}^{\text{hol}}(U)$ we have that $f+g, -f \in \mathcal{O}^{\text{hol}}(U)$, showing that it is an abelian group. Similarly, for $f, g \in \mathcal{O}^{\text{hol},\times}(U)$ we have $f \cdot g, 1/f \in \mathcal{O}^{\text{hol},\times}(U)$. Moreover, for $U \subseteq V \subseteq \mathbb{C}$ we have restriction maps $\mathcal{O}^{\text{hol}}(V) \to \mathcal{O}^{\text{hol}}(U), f \mapsto f|_U$, compatible under compositions (and defining group homomorphisms), which make $\mathcal{O}^{\text{hol}}$ into a presheaf.

For the sheaf axiom, given a cover $\{U_i : i \in I\}$ of some open set U and holomorphic functions f_i on U_i which agree on overlaps, there is a unique function f on U restricting to f_i on U_i. Since being holomorphic can be checked locally at each point, and since the f_i are holomorphic by assumption, also f is holomorphic.

b) The group homomorphism property follows from $\exp(f+g) = \exp(f) \cdot \exp(g)$ converting the addition operation in $\mathcal{O}^{\text{hol}}$ to the multiplication operation in $\mathcal{O}^{\text{hol},\times}$. Compatibility with restrictions is also clear. This then also implies that $\exp_0$ is well-defined: assume $[(U, f)] = [(V, g)] \in \mathcal{O}_0^{\text{hol}}$, then there exists $0 \in W \subseteq U \cap V$ with $f|_W = g|_W$. But then $[(U, \exp(f))] = [(V, \exp(g))] \in \mathcal{O}_0^{\text{hol},\times}$ since $\exp(f)|_W = \exp(g)|_W$.

c) Given f, g holomorphic functions on some open neighborhoods of 0 such that they have the same restriction on some smaller neighborhood W of 0, they have the same derivatives $f^{(k)}(0)$ since taking the derivative is a local operation. This shows that the map is well-defined.

From complex analysis, we know that for f holomorphic around 0, the Taylor expansion $T(f)$ is a power series with a positive radius $r_f > 0$ of convergence and $f(z) = T(f)(z)$ for $z \in B_{r_f}(0)$. This proves injectivity: for $[(U, f)] \in \mathcal{O}_0^{\text{hol}}$ with $T(f) = 0$ we have

$$[(U, f)] = [(B_{r_f}(0) \cap U, f|_{B_{r_f}(0) \cap U})] = [(B_{r_f}(0) \cap U, 0)] \,.$$

Similarly, the image of T is precisely the set $\mathbb{C}\{\{z\}\}$ of formal power series that have a positive radius of convergence. As T is injective, we thus have $\mathcal{O}_0^{\text{hol}} \cong \mathbb{C}\{\{z\}\}$.

d) Unless $U = \emptyset$, the map $\exp_U$ is never injective, since $\exp(f + 2\pi i) = \exp(f)$. For $U = B_1(0)$ the map $\exp_U$ is surjective: let $g : B_1(0) \to \mathbb{C}^*$ be a nowhere-zero holomorphic function. Choose some logarithm $\log(g(0))$, which is unique up to a multiple of $2\pi i$. Then the function $f : B_1(0) \to \mathbb{C}$ given by

$$f(z) = \log(g(0)) + \int_0^z \frac{g'(w)}{g(w)} dw$$

is holomorphic and satisfies $\exp(f) = g$. Here we use that $B_1(0)$ is simply connected.

On the other hand, for $U = \mathbb{C}^*$ and $g(z) = z \in \mathcal{O}^{\text{hol},\times}(U)$ there exists no f with $g = \exp(f)$. Any such f be an inverse to the exp-function and thus would have to satisfy

$$f'(z) = \frac{1}{(\exp')(f(z))} = \frac{1}{\exp(f(z))} = \frac{1}{z}$$

by the formula for the derivative of an inverse function. But $1/z$ has no primitive on $\mathbb{C}^*$ since the path integral of $1/z$ along the unit circle is $2\pi i \neq 0$.

e) One has $\ker(\exp_0) = 2\pi i \cdot \mathbb{Z}$, sitting inside $\mathcal{O}_0^{\mathrm{hol}}$ as the set of integer multiples of the constant function $z \mapsto 2\pi i$. On the other hand, adapting the argument of surjectivity of $\exp_{B_1(0)}$ one can prove that $\exp_0$ is surjective. This shows that $\mathcal{O}_0^{\mathrm{hol},\times} = \mathbb{R}\{\{z\}\}/2\pi i\mathbb{Z}$. This group can be identified with the set of convergent power series with non-zero constant coefficient.

f) No, the map T is no longer injective: the smooth function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \exp(-1/x^2)$ extends to 0 via $f(0) = 0$. The resulting smooth function satisfies $f^{(k)}(0) = 0$ for all k, so $T(f) = 0$. But f is nonzero restricted to any open subset of 0, so

$$0 \neq [(\mathbb{R}, f)] \in \ker(T) \subseteq \mathcal{C}_0^\infty \, .$$