Presence Sheet 4

Exercise 1. A *linear algebraic group* is a tuple (G, m, i, e) of an affine variety G, morphisms

$$m : G \times G \to G \text{ and } i : G \to G$$

and a point $e \in G$ such that

$$\begin{aligned} m(m(g_1, g_2), g_3) &= m(g_1, m(g_2, g_3)) \\ m(e, g) &= m(g, e) = g \\ m(g, i(g)) &= m(i(g), g) = e \end{aligned}$$

for all $g, g_1, g_2, g_3 \in G$. We think of $m(g, h) = g \circ h$ as the group operation, $e \in G$ as the neutral element of the group and $i(g) = g^{-1}$ as the inverse element in the group.

Show that the following are examples of linear algebraic groups (provide the full data (G, m, i, e) above, show that m, i are morphisms and check as many of the properties as you find interesting):

a) $\mathbb{G}_a = \mathbb{A}^1$ with addition $+$

b) $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ with multiplication $\cdot$

c) $\mu_2 = \{1, -1\}$ with multiplication $\cdot$

d) $\mathrm{GL}_n = \{A \in \mathrm{Mat}(n \times n, K) : A \text{ invertible}\}$ with matrix multiplication
Hint: If you are stuck, you can look up the "adjugate matrix" on wikipedia.

Solution.

a) We have $m(x, y) = x + y$ and $i(x) = -x$ are morphisms since they are polynomial in the coordinates x, y and x on $\mathbb{A}^1 \times \mathbb{A}^1$ and $\mathbb{A}^1$. The neutral element is $e = 0 \in \mathbb{A}^1$.

b) We have $m(x, y) = x \cdot y$ and $i(x) = 1/x$ are morphisms since they are regular functions on $(\mathbb{A}^1 \setminus \{0\})^2$ and $\mathbb{A}^1 \setminus \{0\}$ with image in $\mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1$. The neutral element is $e = 1 \in \mathbb{A}^1 \setminus \{0\}$.

c) We have that $\mu_2 \subseteq \mathbb{G}_m$ is a closed subvariety and the restrictions of m, i from $\mathbb{G}_m$ to $\mu_2 \times \mu_2$ and μ_2 have image in μ_2. Thus they give rise to morphisms, and $e = 1$ is contained in μ_2 as well.
Note: μ_2 is an example of a *closed algebraic subgroup* of $\mathbb{G}_m$.

d) The composition map $m(A, B) = A \cdot B$ is polynomial and hence a morphism. To see that the inverse map $i(A) = A^{-1}$ is an algebraic morphism, we have to show that the (i,j)-entry of $A^{-1} \in \mathrm{Mat}(n \times n, K) = \mathbb{A}^{n^2}$ is a regular function on GL_n. But by linear algebra, this entry is given by

$$\frac{(-1)^{i+j}}{\det(A)} \cdot M_{j,i}$$

where $M_{i,j}$ is the determinant of the matrix obtained from A by deleting the i-th row and j-th column. Since $\mathrm{GL}_n = D(\det) \subseteq \mathrm{Mat}(n \times n, K)$, this is indeed a regular function.

Exercise 2. In this exercise, we want to show the following nice topological property of linear algebraic groups:

Proposition Any connected linear algebraic group G is irreducible.

a) Let X, Y be affine varieties and $y_0 \in Y$. Show that the constant map $X \to Y, x \mapsto y_0$ is a morphism.
Bonus challenge: Show the same thing for X, Y prevarieties!

b) Show that for $h \in G$ the *left-translation*

$$t_h : G \to G, g \mapsto m(h,g)$$

is an isomorphism.

c) Show that for any two points $p, q \in G$ there is an isomorphism $\varphi : G \to G$ with $\varphi(p) = q$.

d) Let X be a connected topological space with irreducible decomposition $X = X_1 \cup \ldots \cup X_n$ with $n \geq 2$. Show that there exist

- a point $p \in X$ lying on a unique (i.e. exactly one) irreducible component X_i,
- a point $q \in X$ lying on at least two irreducible components

e) Prove the proposition above.

Solution.

a) For $Y \subseteq \mathbb{A}^m$, the coordinates of the map $x \mapsto y_0$ are constant functions, and thus regular on X. By our criterion from the lecture, this proves that the map $X \to Y$ is a morphism.
Bonus challenge: Cover X by affine varieties X_i and let $U \subseteq Y$ be an affine open containing y_0. By the proof above, all functions $X_i \to U, x \mapsto y_0$ are morphisms, and they agree on overlaps $X_i \cap X_j$. Thus they glue to a unique morphism $X \to U$, which is given by $x \mapsto y_0$. Composing this with the inclusion morphism $U \to Y_0$ we obtain the desired morphism $X \to Y$.

b) To see that the map t_h is a morphism, first note that the map

$$(h, \mathrm{id}_G) : G \to G \times G, g \mapsto (h,g)\,,$$

is a morphism by the universal property of the product $G \times G$ (since both the constant map $G \mapsto G, g \mapsto h$ and the identity are morphisms). Then t_h is the composition $t_h = m \circ (h, \mathrm{id}_G)$. By the properties of linear algebraic groups, the inverse of t_h is given by $t_{i(h)}$.

c) The desired isomorphism is $\varphi = t_{m(q,i(p))}$ since

$$t_{m(q,i(p))}(p) = m(m(q, i(p)), p) = m(q, m(i(p), p)) = m(q, e) = q\,.$$

d) To find p, note that the inclusion $Y := (X_1 \cap X_2) \cup \ldots \cup (X_1 \cap X_n) \subseteq X_1$ must be strict, since otherwise the irreducible set X_1 has a finite cover by strict closed subsets, a contradiction. Take p any point of $X_1 \setminus Y$.

Note: The above property (irreducible spaces have no finite cover by strict closed subsets) follows from the definition of irreducibility by an induction argument!

To find q, note that X_1 must intersect one of the components $X_2, \ldots, X_n$, since otherwise $X = X_1 \sqcup (X_2 \cup \ldots \cup X_n)$ is a decomposition into disjoint closed sets, a contradiction to X being connected. We can take q to be an intersection point of X_1 with $X_2 \cup \ldots \cup X_n$.

e) Assume $X = G$ was connected but not irreducible. By part d) we find a point $p \in G$ lying on exactly one irreducible component, and a point $q \in G$ lying on at least two. But by part c) there exists an isomorphism $\varphi : G \to G$ sending p to q. However, the map φ then induces a bijection from the irreducible components of G to themselves, sending those components containing p to those containing q. Since these sets don't have the same cardinality, this gives a contradiction.