Presence Sheet 5

Exercise 1. Let $0 \neq f \in K[x_0, x_1]$ be a nonzero homogeneous polynomial of degree $d \geq 0$.

a) Let $g(x_1) = f(1, x_1)$ be the dehomogenization of f. Show that

$$f(x_0, x_1) = x_0^d \cdot g(x_1/x_0).$$

b) Show that f has a decomposition

$$f = (b_1 x_0 - a_1 x_1) \cdot (b_2 x_0 - a_2 x_1) \cdots (b_d x_0 - a_d x_1)$$

into linear factors.

c) Show that the vanishing set of f is given by

$$V(f) = \{(a_1 : b_1), \ldots, (a_d : b_d)\} \subseteq \mathbb{P}^1 .$$

We say that the $(a_i : b_i)$, counted with multiplicity, are the zeros of f on $\mathbb{P}^1$.
Note: These multiplicities sum to the degree d of the polynomial.

Solution.

a) The two sides of the formula are linear in f, so it suffices to check them for the monomials $f = x_0^i x_1^{d-i}$ whose linear combinations give all homogeneous degree d polynomials in x_0, x_1. For these the formula is obvious since $x_0^d (x_1/x_0)^{d-i} = x_0^i x_1^{d-i}$.

b) Let $g = f^i \in K[x_1]$ be the dehomogenization of f and let d' be the degree of g. Then since K is algebraically closed, the polynomial g decomposes into a product

$$g(x_1) = \lambda \cdot (x_1 - z_1) \cdot (x_1 - z_2) \cdots (x_1 - z_{d'})$$

Plugging in the formula from part a) we thus have

$$\begin{aligned} f &= x_0^d \cdot \lambda \cdot (\frac{x_1}{x_0} - z_1) \cdot (\frac{x_1}{x_0} - z_2) \cdots (\frac{x_1}{x_0} - z_{d'}) \\ &= x_0^{d-d'} \cdot (\lambda x_1 - \lambda z_1 x_0) \cdot (x_1 - z_2 x_0) \cdots (x_1 - z_{d'} x_0) \, , \end{aligned}$$

which (basically) has the desired shape.

c) The product is zero if and only if one of the factors is zero. But these are the determinants of the matrices $\begin{pmatrix} a_i & b_i \\ x_0 & x_1 \end{pmatrix}$ so this happens if and only if the vectors (x_0, x_1) and (a_i, b_i) are linearly dependent. This in turn is equivalent to $(x_0 : x_1) = (a_i : b_i) \in \mathbb{P}^1$.

Exercise 2. A homogeneous polynomial $f \in K[x_0, x_1]$ of degree $d \geq 0$ is given by

$$f = f_c = c_0 x_0^d + c_1 x_0^{d-1} x_1 + \ldots + c_{d-1} x_0 x_1^{d-1} + c_d x_1^d \,.$$

In the following we identify the space Poly_d of such nonzero polynomials up to scaling with $\mathbb{P}^d$ by sending the class $[f_c]$ of the polynomial f_c to the vector $c = (c_0 : c_1 : \ldots : c_d) \in \mathbb{P}^d$.

For the following sets, decide if they are open, closed or not-well-defined in $\mathrm{Poly}_d = \mathbb{P}^d$, and in the first two cases compute their dimension (assume $d \geq 1$ for simplicity).

a) $A = \{[f] \in \mathrm{Poly}_d : f(p_0) = 0 \text{ for } p_0 = (1:0) \in \mathbb{P}^1\}$

b) $B = \{[f] \in \mathrm{Poly}_d : f(p_0) = f(p_1) \text{ for } p_0 = (1:0), p_1 = (0:1) \in \mathbb{P}^1\}$

c) $C = \{[f] \in \mathrm{Poly}_d : \text{all zeros of } f \text{ have multiplicity } 1\}$

Bonus exercise (optional; guess an answer - proof needs tools we'll not discuss):

d) $D = \{[f] \in \mathrm{Poly}_d : f \text{ has a zero of order at least } 3\}$

Solution.

a) Condition $f(p_0) = 0$ for $f = f_c$ is equivalent to $c_0 = 0$, so $A = V(p_0) \subseteq \mathrm{Poly}_d = \mathbb{P}^d$ is closed. Its cone is given by $C(A) = \{0\} \times \mathbb{A}^d \subseteq \mathbb{A}^d$, so its dimension is $\dim A = \dim C(A) - 1 = d - 1$.

b) It's not well-defined to compare the values of f at p_0, p_1: e.g. $f(x_0, x_1) = x_0 + x_1$ has $f(0,1) = f(1,0) = 1$, but choosing a different representative $p_1 = (0:2) \in \mathbb{P}^1$ we have $f(0,2) \neq f(1,0)$.

c) There is a homogeneous discriminant polynomial $\mathrm{Disc}(f_c)$ which vanishes if and only if f has a zero of higher multiplicity. Thus $C = D(\mathrm{Disc}(f_c))$ is open (and non-empty) in $\mathbb{P}^d$ and thus of dimension d.

d) The set D is the image of the map

$$P : \mathrm{Poly}_1 \times \mathrm{Poly}_{d-3} \to \mathrm{Poly}_d, ([f],[g]) \mapsto [f^3 \cdot g] \,.$$

Since the domain of this map is irreducible and complete (see material of current week), the image D of this map is irreducible and closed. Moreover, the map P is one-to-one over an open subset of its image. By results on dimension theory we did not cover in class (see e.g. [Vakil, Corollary 12.4.2]) this implies that $\dim D = \dim \mathrm{Poly}_1 \times \mathrm{Poly}_{d-3} = d - 2$. This might agree with the intuition that imposing a triple zero is a codimension 2 condition.

Exercise 3. An *effective divisor of degree* d on $\mathbb{P}^1$ is a formal linear combination $D = m_1 p_1 + \ldots + m_k p_k$ of finitely many points $p_i \in \mathbb{P}^1$ with $m_i \in \mathbb{N}$ such that $m_1 + \ldots + m_k = d$. E.g. examples of effective divisors of degree 3 are:

$$D_1 = (0:1) + (1:1) + (1:0) \text{ and } D_2 = 2 \cdot (1:2) + (1:3) \,. \tag{1}$$

Let Eff_d be the set of such effective divisors.

a) Show that the map

$$\Psi : \mathrm{Eff}_d \to \mathrm{Poly}_d \cong \mathbb{P}^d, D = \sum_{i=1}^{k} m_i(a_i : b_i) \mapsto \left[\prod_{i=1}^{k}(b_i x_0 - a_i x_1)^{m_i}\right] \tag{2}$$

is well-defined and bijective. Thus we can interpret $\mathbb{P}^d$ as the set of effective divisors of degree d on $\mathbb{P}^1$.

b) What are the images of D_1, D_2 from (1) under Ψ?

Solution.

a) The map Ψ is well-defined since changing the representatives of the points $(a_i : b_i)$ in D by scaling also scales the resulting polynomial representing $\Psi(D)$. It is injective since we can recover D as the sum of zeros of $\Psi(D)$, counted with multiplicities. On the other hand it's surjective by Exercise 1 above.

b) These images are given by

$$\Psi(D_1) = [x_0 x_1 (x_0 - x_1)] \text{ and } \Psi(D_2) = \left[(2x_0 - x_1)^2 \cdot (3x_0 - x_1)\right].$$