Presence Sheet 7

On this exercise sheet, we'll talk a bit about the topology of algebraic varieties (and work over the field $K = \mathbb{C}$ of the complex numbers everywhere). If X is a topological space, its *Euler characteristic* is defined as the alternating sum

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}} H_i(X(\mathbb{C}), \mathbb{Q})$$

of the dimensions of its homology groups (where we write $X(\mathbb{C})$ for the space X with the complex topology).

However, even without knowing anything about homology groups (and only using the Zariski topology), you can do the entire sheet below just using the following three properties of the Euler characteristic of (complex) algebraic varieties:

(A) If X is a variety which can be written as a disjoint union $X = X_1 \sqcup \ldots \sqcup X_m$ of finitely many locally closed[1] sets $X_i \subseteq X$, then $\chi(X) = \chi(X_1) + \ldots + \chi(X_m)$.

(B) If $\pi : X \to Y$ is a morphism, such that all fibers $X_q = \pi^{-1}(q)$ have the same Euler characteristic $\chi(X_q) = d$ for $q \in Y$, then $\chi(X) = d \cdot \chi(Y)$.

(C) We have $\chi(\{pt\}) = \chi(\mathbb{A}^1) = 1$.

Exercise 1. (Basic spaces)

a) Calculate $\chi(\mathbb{P}^1)$ and $\chi(\mathbb{A}^1 \setminus \{0\})$.

b) Show that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$ for X, Y algebraic varieties.

c) Calculate $\chi(\mathbb{A}^n)$.

d) Calculate $\chi(\mathbb{P}^n)$.

Hint: Start with $n = 2$ in c) and d) if you are stuck.

Solution.

a) We have that $\{(0:1)\} \subseteq \mathbb{P}^1$ is closed with complement $\mathbb{A}^1$, and so we get a locally closed decomposition $\mathbb{P}^1 = \mathbb{A}^1 \sqcup \{(0:1)\}$. Using the properties of Euler characteristics, we have

$$\chi(\mathbb{P}^1) \overset{(A)}{=} \chi(\mathbb{A}^1) + \chi(\{(0:1)\}) \overset{(C)}{=} 1 + 1 = 2\,.$$

[1] Recall that a set $S \subseteq X$ is locally closed if it can be written as the intersection $S = U \cap C$ of a Zariski open set $U \subseteq X$ and a Zariski closed set $C \subseteq X$.

On the other hand $\mathbb{A}^1 = (\mathbb{A}^1 \setminus \{0\}) \sqcup \{0\}$ which proves

$$\underbrace{\chi(\mathbb{A}^1)}_{=1} = \chi(\mathbb{A}^1 \setminus \{0\}) + \underbrace{\chi(\{0\})}_{=1}$$

and so $\chi(\mathbb{A}^1 \setminus \{0\}) = 0$.

b) Let $\pi : X \times Y \to Y$ be the projection morphism. For any $q \in Y$ we have $\pi^{-1}(\{q\}) = X \times \{q\} \cong X$ has Euler characteristic $d = \chi(X)$. So by property (B) we have $\chi(X \times Y) = d \cdot \chi(Y) = \chi(X) \cdot \chi(Y)$.

c) Applying part b) we have $\chi(\mathbb{A}^2) = \chi(\mathbb{A}^1) \cdot \chi(\mathbb{A}^1) = 1 \cdot 1 = 1$ and by induction $\chi(\mathbb{A}^n) = \chi(\mathbb{A}^{n-1}) \cdot \chi(\mathbb{A}^1) = 1$.

d) We have $\mathbb{P}^n = \mathbb{A}^n \sqcup V(x_0)$ with $V(x_0) \cong \mathbb{P}^{n-1}$. Thus

$$\chi(\mathbb{P}^n) = \chi(\mathbb{A}^n) + \underbrace{\chi(\mathbb{P}^{n-1})}_{=n \text{ by induction}} = n + 1\,,$$

where we can use e.g. $\chi(\mathbb{P}^0) = \chi(\{\mathrm{pt}\}) = 1$ as the induction start.

Exercise 2. (Fancier spaces)

a) Calculate $\chi(X)$ for $X = V(x_1 x_2) \subseteq \mathbb{A}^2$.

b) Let $Q_n \subseteq \mathbb{P}^n$ denote an irreducible quadric hypersurface. Calculate $\chi(Q_2)$ for *all* such hypersurfaces and $\chi(Q_3)$ for *one* such hypersurface. What happens for a *reducible* quadric hypersurface in $\mathbb{P}^2$?

Solution.

a) We have $X = V(x_1) \cup V(x_2)$ with $V(x_1) \cong V(x_2) \cong \mathbb{A}^1$ and $V(x_1) \cap V(x_2) = \{(0,0)\}$. Thus we get a locally closed decomposition $X = (\mathbb{A}^1 \setminus \{0\}) \sqcup \{(0,0)\} \sqcup (\mathbb{A}^1 \setminus \{0\})$ and so $\chi(X) = 0 + 1 + 0 = 1$.

b) We have seen that any irreducible conic Q_2 in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ (via projection from a point on Q_2) so $\chi(Q_2) = \chi(\mathbb{P}^1) = 2$. On the other hand, we have seen that $Q_3 = V(x_0 x_3 - x_1 x_2) \subseteq \mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via the Segre embedding. Thus in this case $\chi(Q_3) = \chi(\mathbb{P}^1) \times \chi(\mathbb{P}^1) = 2 \cdot 2 = 4$ using Exercise 1, part b).
Note: In fact *any* such irreducible quadric hypersurface Q_3 has Euler characteristic 4 (this follows e.g. by Ehrensmann's theorem over the connected moduli space of quadric hypersurfaces, using methods we have not yet seen in class).

Any reducible quadric Q_2 is cut out by a reducible quadric polynomial which decomposes as the product of two linear polynomials (which are distinct, since otherwise their vanishing set would not be a quadric but just a plane). Thus $Q_2 = L_1 \cup L_2$ is the union of two lines, meeting in a single point p. Thus

$$\chi(Q_2) = \chi(\underbrace{L_1 \setminus \{q\}}_{\cong \mathbb{P}^1 \setminus \{\mathrm{pt}\} = \mathbb{A}^1}) + \chi(\{q\}) + \chi(\underbrace{L_2 \setminus \{q\}}_{\cong \mathbb{P}^1 \setminus \{\mathrm{pt}\} = \mathbb{A}^1}) = 1 + 1 + 1 = 3\,.$$

Exercise 3. (Fanciest spaces)

a) Compute the Euler characteristic $\chi(G(2,4))$.
Hint: Look at [Gathmann, Remark 8.20].
Bonus: Can you find the formula for $\chi(G(k,n))$?

b) Let $0 \neq f \in K[x_1]$ be a homogeneous polynomial of degree $2g+2$ for some $g \in \mathbb{N}$ which has only simple zeros. Consider the affine curve

$$C = \{(x_1, x_2) \in \mathbb{A}^2 : x_2^2 = f(x_1)\} \subseteq \mathbb{A}^2$$

The variety C is called an *affine hyperelliptic curve of genus g*. Show that $\chi(C) = -2g$.
Hint: Try to find a morphism from C to a simpler space, which has finite fibers.

Solution.

a) We follow the cited Remark 8.20 in [Gathmann]: any 2-plane in K^4 can be generated by the rows of a 2×4-matrix M. These generators are unique up to row operations. On the other hand, the matrix M has a unique row-reduced echelon form $\widetilde{M}$. The possible shapes of this form $\widetilde{M}$ are

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}, \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

where all entries $*$ represent arbitrary numbers in K. Then it's true that each subset of $G(2,4)$ representing a given shape of $\widetilde{M}$ is locally closed, and isomorphic to an affine space $\mathbb{A}^n$ for n the number of $*$-entries in $\widetilde{M}$. Each of these sets has Euler characteristic 1, so using property (A) we have

$$\chi(G(2,4)) = \#\{\text{shapes of } \widetilde{M}\} = 6\,.$$

Using the same argument for $G(k,n)$, we note that the shapes of the row-reduced echelon form $\widetilde{M}$ are precisely specified by choosing the k columns in the matrix where we write a 1. There are $\binom{n}{k}$ such choices, so $\chi(G(k,n)) = \binom{n}{k}$.

b) Consider the projection morphism

$$\pi : C \to \mathbb{A}^1, (x_1, x_2) \mapsto x_1\,.$$

Let's calculate the number of preimages $\pi^{-1}(q)$ for $q \in \mathbb{A}^1$.

To do this, let $V(f) = \{q_1, \ldots, q_{2g+2}\} \subseteq \mathbb{A}^1$ be the $2g+2$ zeros of f (here we use that all zeros of f have multiplicity exactly 1). Then since the equation $x_2^2 = r$ has two solutions in $\mathbb{C}$ for $r \neq 0$ and one solution otherwise, we have:

$$\#\pi^{-1}(q) = \begin{cases} 2 & \text{if } q \in U = \mathbb{A}^1 \setminus V(f) \\ 1 & \text{if } q \in V(f) = \{q_1, \ldots, q_{2g+2}\}\,. \end{cases} \tag{1}$$

We want to apply property (B) to calculate the Euler characteristic, but the problem is, that the number of preimages is not the same everywhere. The final trick is to just decompose both $\mathbb{A}^1$ and C into locally closed pieces, on which the degree is constant: we have that the maps

$$\pi^{-1}(U) \xrightarrow{\pi} U \text{ and } \pi^{-1}(V(f)) \xrightarrow{\pi} V(f)$$

are morphisms of varieties, with $2, 1$ preimages at any point, respectively. Thus

$$\chi(C) \overset{(A)}{=} \chi(\pi^{-1}(U)) + \chi(\pi^{-1}(V(f))) \overset{(B)}{=} 2 \cdot \chi(U) + 1 \cdot \chi(V(f))\,.$$

We have that $V(f)$ is the disjoint union of $2g+2$ points, so $\chi(V(f)) = 2g+2$. On the other hand, if one adds this number to the Euler characteristic $\chi(U) = \chi(\mathbb{A}^1 \setminus V(f))$ one needs to recover $\chi(\mathbb{A}^1) = 1$. Thus $\chi(U) = 1 - \chi(V(f)) = 1 - (2g+2) = -2g-1$. Plugging this into the formula above, we have

$$\chi(C) = 2 \cdot (-2g-1) + 1 \cdot (2g+2) = -2g\,.$$