Presence Sheet 8

Exercise 1. (Rational maps $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$)

a) Let $f_1, f_2 \in K[x_0, x_1]$ be two non-zero homogeneous polynomials of the same degree $d \geq 0$. Show that the assignment

$$f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1, x \mapsto (f_1(x) : f_2(x)) \tag{1}$$

gives a rational map. What is the open set of $\mathbb{P}^1$ where it is defined?

b) Compute the domain of definition of f for $f_1 = x_0^2 + x_0x_1$ and $f_2 = x_0x_1 + x_1^2$. Show that on its domain of definition, f is given by the identity function (and thus can be extended to all of $\mathbb{P}^1$).

c) Use the idea of the last part to show that any map f as in (1) can be extended to all of $\mathbb{P}^1$.
Hint: It might help to do an induction on the degree d.

d) Let $g \in K[x]$ be a non-constant polynomial with associated distinguished open $D(g) \subseteq \mathbb{A}^1$ and $\widetilde{f} : D(g) \to \mathbb{A}^1$ a morphism. Seeing $D(g) \subseteq \mathbb{A}^1 \subseteq \mathbb{P}^1$ show that there is a rational map $f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ of the form (1) that agrees with $\widetilde{f}$ on an open subset of $\mathbb{P}^1$.

e) Conclude that any rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ can be extended to a morphism $\mathbb{P}^1 \to \mathbb{P}^1$.

Solution.

a) We showed in [Gathmann, Lemma 7.4] that f gives a well-defined morphism on the complement of $V(f_1, f_2)$. This complement is open and non-empty by assumption (since f_1, f_2 are non-zero).

b) The rational map f is undefined exactly on

$$V(x_0^2 + x_0x_1, x_0x_1 + x_1^2) = V(x_0(x_0 + x_1), x_1(x_0 + x_1)) = \underbrace{V(x_0, x_1)}_{=\emptyset} \cup \underbrace{V(x_0 + x_1)}_{=\{(1:-1)\}}.$$

Away from the point $(1 : -1)$ the quantity $x_0 + x_1$ is nonzero, so we can divide both components of f by this and obtain

$$f(x_0 : x_1) = (x_0(x_0 + x_1) : x_1(x_0 + x_1)) = (x_0 : x_1),$$

so indeed f equals the identity there.

c) We prove the statement by induction on d. When $d = 0$ both f_1, f_2 are non-zero constants so f is also constant and defined everywhere.

Assume the statement is proven up to degree $d-1$ and let f_1, f_2 be non-zero homogeneous of degree d. If they do not have a common zero $(a : b)$, then f is defined everywhere and we are done. Otherwise, both f_1, f_2 are divisible by $x_0b - x_1a$ (since we proved that $I(\{(a : b)\}) = \langle x_0b - x_1a\rangle$ on a previous exercise sheet) and away from $(a : b)$ this quantity is nonzero. After dividing, we have

$$f(x) = (f_1(x) : f_2(x)) = (\frac{f_1}{x_0b - x_1a}(x) : \frac{f_2}{x_0b - x_1a}(x))\,.$$

Thus we have reduced the degree of the defining polynomials and by induction we conclude that it extends on all of $\mathbb{P}^1$.

d) We have seen that a morphism $\widetilde{f}$ to $\mathbb{A}^1$ must be given by a regular function on $D(g)$. On the other hand, this function must be given by $\widetilde{f} = h/g^m$ for some $m \in \mathbb{N}$ and $h \in K[x]$. Let $d = \deg(h)$ and $e = \deg(g)$ and set $M = \max(d, em)$, then we define

$$f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1, (x_0 : x_1) \mapsto (x_0^M g(x_1/x_0)^m : x_0^M h(x_1/x_0))$$

On the affine patch U_0 where we set $x_0 = 1$ this agrees with the map $(g(x_1)^m : h(x_1)) = (1 : h(x_1)/g(x_1)^m) = (1 : \widetilde{f}(x_1))$, so indeed $\widetilde{f}$ and f agree where they are defined.

e) Let $F : \mathbb{P}^1 \supseteq U \to \mathbb{P}^1$ be any rational map. If F is constant and equal to $\infty = (0 : 1)$ it can clearly be extended on all of $\mathbb{P}^1$. Otherwise, restricting to $U' = F^{-1}(\mathbb{A}^1)$ we get a morphism $F : U' \to \mathbb{A}^1$. But $U' \subseteq \mathbb{A}^1$ is a non-empty open set and thus isomorphic to $U' = D(g)$ for some g. By part d) the morphism F then agrees with a map f of the form (1). But this can be extended to all of $\mathbb{P}^1$ by part c), finishing the proof.

Exercise 2. (Cremona transformation, see Gathmann Exercise 9.29)
Let $a = (1 : 0 : 0), b = (0 : 1 : 0), c = (0 : 0 : 1)$ be the three coordinate points of $\mathbb{P}^2$ and $U = \mathbb{P}^2 \setminus \{a, b, c\}$. Consider the morphism

$$f : U \to \mathbb{P}^2, (x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1)\,.$$

a) Show that there is no morphism $\mathbb{P}^2 \to \mathbb{P}^2$ extending f.
Hint: For coordinates $(z_0 : z_1 : z_2)$ on the target, calculate $f^{-1}(V(z_i))$.

b) Show that f is dominant and $f \circ f \sim \mathrm{id}_{\mathbb{P}^2}$ as rational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Bonus exercise:

c) Let $\widetilde{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ at $\{a, b, c\}$. Show that f extends to a morphism $\widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$.
Note: In fact it even extends to a *isomorphism* $\widetilde{\mathbb{P}}^2 \to \widetilde{\mathbb{P}}^2$, called the Cremona transformation.

Solution.

a) We have $f^{-1}(V(z_0)) = V(x_1x_2) \cap U$ and similarly for the other coordinates z_1, z_2. If f extended to a morphism $\mathbb{P}^2 \to \mathbb{P}^2$ it would have to send the closure $V(x_1x_2) = \overline{V(x_1x_2) \cap U}$ of this locus to $V(z_0)$. Taking the intersection of all of these, we have that all points in

$$V(x_1x_2) \cap V(x_0x_2) \cap V(x_0x_1) = \{a, b, c\}$$

must be sent to $V(z_0, z_1, z_2) = \emptyset$, giving a contradiction.

b) On the chart $U_0 = \{x : x_0 \neq 0\}$ of the domain, setting $x_0 = 1$, the map is given by

$$f : U_0 \setminus \{0\} \to \mathbb{P}^2, (x_1, x_2) \mapsto (x_1x_2 : x_2 : x_1) = (1 : 1/x_1 : 1/x_2)\,.$$

where the last equality is allowed when $(x_1, x_2) \in (\mathbb{A}^1 \setminus \{0\})^2$. Since the map

$$(\mathbb{A}^1 \setminus \{0\})^2 \to (\mathbb{A}^1 \setminus \{0\})^2, (x_1, x_2) \mapsto (1/x_1, 1/x_2)$$

is an isomorphism, this proves that the image of f contains a non-empty open subset, and thus f is dominant. Composing it with itself we have

$$f(f(x_0 : x_1 : x_2)) = (x_0x_2x_0x_1 : x_1x_2x_0x_1 : x_1x_2x_0x_2) = (x_0 : x_1 : x_2)\,,$$

where the equalities make sense whenever x_0, x_1, x_2 are all nonzero, and where in the last equality we divide by $x_0x_1x_2$ in all components of the vector.

c) As seen above, on the chart U_0 the map is given by

$$f : U_0 \setminus \{0\} \to \mathbb{P}^2, (x_1, x_2) \mapsto (x_1x_2 : x_2 : x_1)$$

Taking the closure inside the blow-up of a (which corresponds to $0 \in U_0$), we have coordinates $((x_1, x_2), (y_1 : y_2))$ with $y_1x_2 = y_2x_1$. In these coordinates we have

$$(x_1x_2 : x_2 : x_1) = (x_1x_2y_1 : x_2y_1 : x_1y_1) = (x_1x_1y_2 : x_1y_2 : x_1y_1) = (x_1y_2 : y_2 : y_1)\,,$$

where in the first equality we multiply all components by y_1, in the second we use $x_1y_2 = x_2y_1$ and in the third we divide all components by x_1 (which is allowed as long as we have $x_1 \neq 0$). This last vector in the equality is a well-defined point in $\mathbb{P}^2$ (since $(y_1, y_2) \neq 0$ implies $(x_1y_2, y_2, y_1) \neq 0$). Thus indeed we can extend f over a (and similarly over b, c).