Presence Sheet 9

Exercise 1. Consider the algebraic variety

$$X = \{M \in \mathrm{Mat}(2 \times 2, K) : \det M = 0\}.$$

a) Show that X is irreducible of dimension 3.

b) Show that the zero matrix $M = 0$ is the only singular point of X.

c) Let $\widetilde{X} = \mathrm{Bl}_0 X$ be the blow-up of X at the origin.

   i) Show that $\widetilde{X}$ is smooth.

   ii) Show that the exceptional locus is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Note: The variety X is called the *cone over the quadric surface*, and the blow-up $\widetilde{X} \to X$ is the *resolution of the conical singularity.*

Solution.

a) Since det is a nonzero irreducible polynomial (e.g. by the Eisenstein criterion), we know that X is an irreducible hypersurface (by Krull's principal ideal theorem), and thus of dimension 3.

b) For $M = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ we have $\det(M) = x_1x_4 - x_2x_3$. Its Jacobi matrix is thus given by

$$J = \left(\frac{\partial \det(M)}{\partial x_i}\right)_{i=1,\dots,4} = \begin{pmatrix} x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}.$$

It vanishes precisely when $x_4 = x_3 = x_2 = x_1 = 0$, so at $M = 0$, thus this is the only singular point by the affine Jacobi criterion.

c) We know that

$$\widetilde{X} \subset \widetilde{\mathbb{A}}^4 = \{(x_1, x_2, x_3, x_4), (y_1 : y_2 : y_3 : y_4) \in \mathbb{A}^4 \times \mathbb{P}^3 : x_iy_j = x_jy_i \ \forall i, j\}.$$

   i) It suffices to check this on the affine patch $U = \{(x, y) \in \widetilde{X} : y_1 \neq 0\}$, with the other affine patches being similar. This patch has coordinates (x_1, y_2, y_3, y_4) with setting $y_1 = 1$ and $x_j = x_1y_j$ for $j = 2, 3, 4$. Thus in these coordinates, the determinant is given by

$$x_1x_4 - x_2x_3 = x_1(x_1y_4) - (x_1y_2)(x_1y_3) = x_1^2(y_4 - y_2y_3).$$

Taking the strict transform means removing the preimage of the origin (which in U is given by $V(x_1)$) and then taking the Zariski closure. This removes $V(x_1^2) = V(x_1)$ from the above equation, so

$$\widetilde{X} \cap U \cong V(y_4 - y_2 y_3) \cong \mathbb{A}^3$$

where the last isomorphism sends $(x_1, y_2, y_3) \in \mathbb{A}^3$ to $(x_1, y_2, y_3, y_2 y_3) \in U$. Since $\mathbb{A}^3$ is smooth, this finishes the proof.

ii) Intersecting $\widetilde{X}$ with the exceptional set of the blow-up $\widetilde{\mathbb{A}}^4 \to \mathbb{A}^4$ in the chart U above corresponds to setting $x_1 = 0$, which does not actually affect the equation, so on U the exceptional locus is just $V(y_4 - y_2 y_3) \subseteq \{(1 : y_2 : y_3 : y_4) \in \mathbb{P}^3\}$. Taking the projective closure corresponds to homogenizing this equation, giving us $Y = V_p(y_1 y_4 - y_2 y_3) \subseteq \mathbb{P}^3$ where we use coordinates y_1, y_2, y_3, y_4 on $\mathbb{P}^3$. To make sure we do not miss any other parts of the exceptional locus, we could repeat this calculation on the other charts of $\widetilde{\mathbb{A}}^4$ and would always exactly obtain the different affine charts of $Y \subseteq \mathbb{P}^3$. We saw that this quadric Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (via the Segre embedding), finishing the proof.

Exercise 2. Recall from Presence Sheet 4 the notion of a linear algebraic group $G = (G, m, i, e)$. Show that any linear algebraic group is smooth.
Hint: Combine [Gathmann, Remark 10.20] with the strategies and results applied on Presence Sheet 4.

Solution. The Remark from the above hint explains the result on generic smoothness, showing that the smooth locus of G is dense, so in particular there exists a smooth point $q \in G$. Take any other point $p \in G$, then we saw on the above presence sheet that there is an isomorphism $\varphi : G \to G$ sending q to p (by left-translation with pq^{-1}). We saw that the tangent space is an intrinsic invariant (e.g. can be calculated from the local ring) and thus an isomorphism sends smooth points to smooth points. Since q is smooth we thus also have that p is smooth.