Exercise Sheet 1

Exercise 1. Consider the ideal $J = \langle x_i - x_1 x_{i-1} : i = 2, \ldots, n \rangle \trianglelefteq K[x_1, \ldots, x_n]$ and let $X = V(J) \subseteq \mathbb{A}^n$ be its vanishing locus.

(a) Show that the projection

$$\pi : X \to \mathbb{A}^1, (x_1, \ldots, x_n) \mapsto x_1$$

is a bijection. Calculate the inverse map $\pi^{-1} : \mathbb{A}^1 \to X$ parameterizing X.

(b) Show that J is a prime ideal.
Hint: Calculate the quotient $K[x_1, \ldots, x_n]/J$.

(c) Conclude that $J = I(X)$ and compute the coordinate ring $A(X)$.

Exercise 2. Determine the radical of the ideal $J = \langle x_1^3 - x_2^6, x_1 x_2 - x_2^3 \rangle \subseteq \mathbb{C}[x_1, x_2]$.
Hint: The Nullstellensatz might be useful here.

Exercise 3. Let $X \subset \mathbb{A}^n$ be an affine variety. Show that the coordinate ring $A(X)$ is a field if and only if X is a single point.

Exercise 4. Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes.

(a) Compute generators for the ideal $I(X)$.

(b) Show that $I(X)$ cannot be generated by fewer than three elements.