Exercise Sheet 10

Exercise 1. As in Sheet 9, Exercise 3, let $U \subseteq \mathbb{P}^{\binom{4+5}{4}-1} = \mathbb{P}^{125}$ be the set of all smooth (3-dimensional) hypersurfaces of degree 5 in $\mathbb{P}^4$.

a) Using the Jacobi criterion, show that the incidence correspondence

$$M := \{(X, L) \in U \times G(2,5) : L \text{ is a line contained in } X\}$$

is smooth of dimension 125, i.e., of the same dimension as U.
Hint: Reduce to showing smoothness when $L = \mathrm{Lin}(e_1, e_2)$. Then for $X = V_p(f_c)$ with

$$f_c = c_0 x_0^5 + c_1 x_0^4 x_1 + \ldots + c_5 x_1^5 + c_6 x_0^4 x_2 + \ldots$$

apply the Jacobi criterion for the partial derivatives along the variables $c_0, \ldots, c_5$.

b) Although (a) suggests that a smooth hypersurface of degree 5 in $\mathbb{P}^4$ contains only finitely many lines, show that the Fermat hypersurface $V(x_0^5 + \cdots + x_4^5) \subset \mathbb{P}^4$ contains infinitely many lines.
Hint: Consider lines of the form $L = \{(a_0 s : a_1 s : a_2 t : a_3 t : a_4 t) : (s : t) \in \mathbb{P}^1\}$ for suitable $a_0, \ldots, a_4 \in \mathbb{C}$.)

Exercise 2. Find an example of the following, or prove that it does not exist:

a) an irreducible affine scheme $\operatorname{Spec} R$ such that R is not an integral domain;

b) a point of $\operatorname{Spec}(\mathbb{R}[x_1, x_2]/\langle x_1^2 + x_2^2 + 1\rangle)$ with residue field $\mathbb{R}$;

c) two affine schemes $\operatorname{Spec} R$ and $\operatorname{Spec} S$ with $R \subseteq S$ and $\dim(\operatorname{Spec} R) > \dim(\operatorname{Spec} S)$;

d) an affine scheme of dimension 1 with exactly two points.

Exercise 3.

a) Let $R = A(X)$ be the coordinate ring of an affine variety X over an algebraically closed field. Show that the set of all closed points is dense in $\operatorname{Spec} R$ (which means by definition that every non-empty open subset of $\operatorname{Spec} R$ contains a closed point).

b) In contrast to (a), however, show by example that on a general affine scheme the set of all closed points need not be dense.