Exercise Sheet 11

Exercise 1. Let R be a ring. Prove that the affine scheme $\operatorname{Spec} R$ is disconnected if and only if $R \cong S \times T$ for two non-zero rings S and T.

Exercise 2. For $n \in \mathbb{N}_{>0}$, an n-fold point over an algebraically closed field K is an affine scheme $\operatorname{Spec} R$ that contains only one point, and such that R is a K-algebra of vector space dimension n over K.

a) Show that every single point over K is isomorphic to $\operatorname{Spec} K$.

b) Show that every double point over K is isomorphic to $\operatorname{Spec} K[x]/\langle x^2 \rangle$.

c) Is part b) correct without the assumption that K is algebraically closed?

d) Find two non-isomorphic triple points over an algebraically closed field K. Here we mean: there is no isomorphism $\operatorname{Spec} R_1 \xrightarrow{\sim} \operatorname{Spec} R_2$ coming from a K-algebra homomorphism $R_2 \to R_1$. Can you describe them geometrically?

Exercise 3.

a) Let K be a field. Show that $\operatorname{Spec} K[x]/\langle x^3 - x^2 \rangle \cong \operatorname{Spec} K[x]/\langle x^2 \rangle \sqcup \operatorname{Spec} K$.

b) For $R = K[x, y]$ calculate the scheme-theoretic intersection $X_1 \cap X_2$ of the two affine subschemes

$$X_1 = \operatorname{Spec} R/\langle x^2 + y^2 - 1 \rangle \to \operatorname{Spec} R \text{ and } X_2 = \operatorname{Spec} R/\langle y - x^2 + 1 \rangle \to \operatorname{Spec} R\,.$$

How many connected components does $X_1 \cap X_2$ have?