Exercise Sheet 12

Exercise 1. Show that for a scheme X the following are equivalent:

a) X is reduced, i.e., for every open subset $U \subset X$ the ring $\mathcal{O}_X(U)$ has no non-zero nilpotent elements.

b) There is an open cover of X by affine schemes $U_i = \operatorname{Spec} R_i$ such that every ring $\mathcal{O}_X(U_i) = R_i$ has no non-zero nilpotent elements.

c) For every point $p \in X$, the local ring $\mathcal{O}_{X,p}$ has no non-zero nilpotent elements.

Exercise 2.

a) Let X be the scheme $\operatorname{Spec} \mathbb{Z}[x,y]/(x^2+y^2-1)$ and $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]$ be the affine line over $\mathbb{Z}$. Provide an explicit isomorphism

$$X \supseteq D(2(y-1)) \simeq D(2(t^2+1)) \subseteq \mathbb{A}^1_{\mathbb{Z}}$$

between open subschemes of X and $\mathbb{A}^1_{\mathbb{Z}}$.
Hint: It is useful to recall the proof of birational equivalence of an irreducible quadric and projective space over an algebraically closed field.

b) What are the $\mathbb{Q}$-points if X i.e. morphisms $\operatorname{Spec} \mathbb{Q} \to X$? Describe them explicitly using the isomorphism above. Use this to describe Pythagorean triples explicitly.

c) How many $\mathbb{F}_p$-points does X have? You can use the fact that the equation $t^2 = -1$ has a solution in $\mathbb{F}_p$ for odd p if and only if $p = 1 \mod 4$.

Exercise 3. Let X be a scheme and Z be a closed subset of the underlying topological space of X. Show that there is a unique closed subscheme Y of X such that its underlying topological space is Z and Y is reduced.

Exercise 4. Recall that for any $\mathbb{F}_p$-algebra R there is a morphism $x \mapsto x^p$ called the Frobenius morphism of R.

a) Let X be a scheme over $\mathbb{F}_p$ i.e. with a morphism $X \to \operatorname{Spec} \mathbb{F}_p$. Show that the structure sheaf takes values in $\mathbb{F}_p$-algebras and the Frobenius morphisms on affine charts glue uniquely to a morphism $\operatorname{Frob}\colon X \to X$ which is called the absolute Frobenius morphism of X.

b) Let X be a variety over $\mathbb{F}_p$. Describe the scheme–theoretic intersection of the graphs of Frob and identity in $X \times X$.