Exercise Sheet 13

Exercise 1.

a) Any morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves of modules on a scheme X determines induced $\mathcal{O}_{X,p}$-module homomorphisms

$$f_p : \mathcal{F}_p \to \mathcal{G}_p, [(U, \phi)] \mapsto [(U, f_U(\phi))]\,,$$

on the stalks for all $p \in X$. Show that f is an isomorphism if and only if all f_p are isomorphisms.

b) Conclude that f is an isomorphism if and only if f is injective and surjective.

Exercise 2. Let $\mathcal{F}$ be a presheaf on a scheme X, and denote by $\theta : \mathcal{F} \to \mathcal{F}^{\text{sh}}$ its sheafification. Prove that any morphism $f' : \mathcal{F} \to \mathcal{G}$ to a sheaf $\mathcal{G}$ factors uniquely through θ, i.e., there is a unique morphism $f : \mathcal{F}^{\text{sh}} \to \mathcal{G}$ with $f' = f \circ \theta$.

Exercise 3. Let X be a topological space and consider the presheaves

$$\mathcal{F}(U) = \{\varphi : U \to \mathbb{R} : \varphi \text{ continuous and bounded}\} \text{ for } U \subseteq X \text{ open}\,,$$
$$\mathcal{C}_X(U) = \{\varphi : U \to \mathbb{R} : \varphi \text{ continuous}\} \text{ for } U \subseteq X \text{ open}\,.$$

a) Give an example of a space X where $\mathcal{F}$ is not a sheaf.

b) Convince yourself that $\mathcal{C}_X$ is a sheaf.

c) Show that the sheafification of $\mathcal{F}$ is given by the sheaf $\mathcal{C}_X$.

d) Let $p : X \to \{\text{pt}\} =: Y$ be the constant map to a point. What is the sheaf $p_*\mathcal{C}_X$?

Note: Here we use sheafification for sheaves on an arbitrary topological space. The definition of sheafification here is just exactly the same as for schemes.

Exercise 4. Let $n \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$. Prove that $\mathcal{O}_{\mathbb{P}^n}(d)^\vee \cong \mathcal{O}_{\mathbb{P}^n}(-d)$.
Note: For inspiration you can re-read the proof from [Gathmann, Example 13.23]. In your proof, you might want to use the presheaf $\mathcal{O}_{\mathbb{P}^n}(d)^{\vee,\text{pre}}$ given by $U \mapsto \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_{\mathbb{P}^n}(d)|_U, \mathcal{O}_U)$.