Exercise Sheet 14

Exercise 1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes.

a) Recall the definition of all the data involved, and then define what the composition $g \circ f : X \to Z$ is. Verify that it satisfies all necessary properties.

b) For $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$ and $Z = \mathrm{Spec}(T)$ with f, g coming from ring homomorphisms $S \to R, T \to S$ the composition $g \circ f$ comes from the composite morphism $T \to S \to R$.

Note: Cynics might say that we should probably have done this in class, and they wouldn't be entirely wrong ...

Exercise 2. Let $\mathcal{F}$ be an invertible sheaf (i.e. locally free of rank 1) on $\mathbb{A}^1_K = \mathrm{Spec} K[x]$ for K a field. Our goal below is to show that $\mathcal{F} \cong \mathcal{O}_{\mathbb{A}^1_K}$ is trivial.

a) Why is $\mathcal{F}$ is of the form $\mathcal{F} = \widetilde{M}$ for M a module over $K[x]$?

b) Let $D(f_i) \subseteq \mathbb{A}^1_K$ be a distinguished open such that there is an isomorphism

$$\varphi_i : \widetilde{R_{f_i}} = \mathcal{O}_{D(f_i)} \xrightarrow{\sim} \mathcal{F}|_{D(f_i)} = \widetilde{M_{f_i}}.$$

Let $\varphi_i(1) = m_i / f_i^{r_i}$ for $m_i \in M$ and $r_i \in \mathbb{N}$. Show that the map

$$s_i : \widetilde{R} \to \widetilde{M} \text{ induced by } R \to M, a \mapsto a m_i$$

induces surjective maps of stalks $s_{i,p} : R_p \to M_p$ for all $p \in D(f_i)$.

c) Show that $\mathcal{F}$ is of the form $\mathcal{F} = \widetilde{M}$ for M a finitely generated module over $K[x]$.
Hint: Choose a finite cover of $\mathbb{A}^1_K$ by sets $D(f_1), \ldots, D(f_n)$ as in the previous part of the exercise and construct a surjection $R^n \to M$.

d) Prove that $\mathcal{F} \cong \mathcal{O}_{\mathbb{A}^1_K}$ is trivial.
Hint: Recall a certain statement about finitely generated modules over principal ideal domains.

Exercise 3. The goal of this exercise is to prove that for K an algebraically closed field, the set of automorphisms of $\mathbb{P}^n_K$ over K is isomorphic to the projective linear group $\mathrm{PGL}(n+1, K)$. The crucial input for the proof will be the following result, which you can use below:

Thm. Any invertible sheaf $\mathcal{L}$ on $\mathbb{P}^n_K$ is of the form $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^n_K}(d)$ for some $d \in \mathbb{Z}$.

a) Show that for $f : X \to Y$ a morphism and $\mathcal{F}, \mathcal{G}$ locally free sheaves on Y, one has $f^*(\mathcal{F} \otimes \mathcal{G}) \cong (f^*\mathcal{F}) \otimes (f^*\mathcal{G})$.
Hint: You can use without proof that it's sufficient to show this for X, Y affine schemes and $\mathcal{F}, \mathcal{G}$ trivial, i.e. direct sums of $\mathcal{O}_Y$.

b) Recall from class that any morphism $f : X \to \mathbb{P}^n_K$ is given by the data of an invertible sheaf $\mathcal{L}_f = f^*\mathcal{O}_{\mathbb{P}^n_K}(1)$ together with sections $s_0, \dots, s_n \in \mathcal{L}(X)$ not vanishing simultaneously anywhere on X. Assume that $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ and $g : \mathbb{P}^n_K \to \mathbb{P}^n_K$ are given by line bundles $\mathcal{L}_f = \mathcal{O}_{\mathbb{P}^n_K}(d_f)$ and $\mathcal{L}_g = \mathcal{O}_{\mathbb{P}^n_K}(d_g)$. Show that we have $d_f, d_g \geq 0$ and the composition $g \circ f$ is given by $\mathcal{L}_{g \circ f} \cong \mathcal{O}_{\mathbb{P}^n_K}(d_f \cdot d_g)$.

c) Conclude that for any isomorphism $f : \mathbb{P}^n_K \xrightarrow{\sim} \mathbb{P}^n_K$ one has $f^*\mathcal{O}_{\mathbb{P}^n_K}(1) \cong \mathcal{O}_{\mathbb{P}^n_K}(1)$.

d) Finish the proof that any automorphism $f \in \mathrm{Aut}_K(\mathbb{P}^n_K, \mathbb{P}^n_K)$ is given by a projective linear map in $\mathrm{PGL}(n+1, K)$.