Exercise Sheet 2

Exercise 1. Let $X \subset \mathbb{A}^n$ be an arbitrary subset. Prove that $V(I(X)) = \overline{X}$.

Exercise 2. (Topology marathon) Let X be a topological space. Show the following statements.

a) If $X = X_1 \cup \ldots \cup X_n$ with X_i Noetherian, then X is Noetherian as well.

b) If $Y \subseteq A \subseteq X$ is closed in the subspace topology of A, then $\overline{Y} \cap A = Y$.

c) A set $A \subseteq X$ is irreducible if and only if $\overline{A}$ is irreducible.

d) If X is irreducible and $U \subseteq X$ is open, then U is irreducible.

e) If $f : X \to Y$ is continuous and X is connected, then so is $f(X)$.

f) If $f : X \to Y$ is continuous and X is irreducible, then so is $f(X)$.

Exercise 3. Calculate the irreducible components of $X = V(J)$ for

$$J = \langle y^2 - x^4, x^2 - 2x^3 - x^2y + 2xy + y^2 - y \rangle \subseteq K[x, y].$$

Hint: The answer depends on the characteristic of K.

Exercise 4. Let $\{U_i : i \in I\}$ be an open cover of a topological space X and assume that $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. Show:

a) If U_i is connected for all i, then so is X.

b) If U_i is irreducible for all i, then so is X.

Exercise 5. Let X be a topological space. Prove:

a) If $\{U_i : i \in I\}$ is an open cover of X, then $\dim X = \sup\{\dim U_i : i \in I\}$.

b) If X is an irreducible affine variety and $U \subseteq X$ a non-empty open subset, then $\dim X = \dim U$.

c) Does the statement from (b) hold more generally for any irreducible topological space?