Exercise Sheet 3

Exercise 1. Let $\varphi, \psi \in \mathcal{F}(U)$ be two sections of a sheaf $\mathcal{F}$ on an open subset U of a topological space X. Show:

a) If φ and ψ agree in all stalks i.e., $[(U, \varphi)] = [(U, \psi)] \in \mathcal{F}_a$ for all $a \in U$ then $\varphi = \psi$.

b) If $\mathcal{F} = \mathcal{O}_X$ is the sheaf of regular functions on an irreducible affine variety X then we can already conclude that $\varphi = \psi$ if we only know that they agree in *one* stalk $\mathcal{F}_a$ for $a \in U$.
Hint: It might help to first cover U by distinguished affine open subsets.

c) For a general sheaf $\mathcal{F}$ on a topological space X the statement of (b) is false.

Exercise 2. Let a be any point on the real line $\mathbb{R}$. For which of the following sheaves $\mathcal{F}$ on $\mathbb{R}$ (with the standard topology) is the stalk $\mathcal{F}_a$ actually a local ring in the algebraic sense (i.e., it has exactly one maximal ideal)?

a) $\mathcal{F}$ is the sheaf of continuous functions;

b) $\mathcal{F}$ is the sheaf of locally polynomial functions.

Exercise 3. Let Y be a non-empty irreducible subvariety of an equidimensional affine variety X and set $U = X \setminus Y$.

a) Assume that $A(X)$ is a unique factorization domain. Show that $\mathcal{O}_X(U) = A(X)$ if and only if $\mathrm{codim}_X Y \geq 2$.

b) Show by example that the equivalence of (a) is in general false if $A(X)$ is not assumed to be a unique factorization domain.
Note: It's pretty hard (but not impossible) to given an example with X irreducible, but feel free to look for a reducible example.

Exercise 4. Let $\mathcal{F}$ be a sheaf on a topological space X and let Y be a non-empty irreducible closed subset of X. We define the *stalk of $\mathcal{F}$ at Y* to be

$$\mathcal{F}_Y := \{(U, \varphi) : U \text{ is an open subset of } X \text{ with } U \cap Y \neq \emptyset \text{ and } \varphi \in \mathcal{F}(U)\} / \sim$$

where $(U, \varphi) \sim (U', \varphi')$ if and only if there is an open set $V \subset U \cap U'$ with $V \cap Y \neq \emptyset$ and $\varphi|_V = \varphi'|_V$. It therefore describes functions in an arbitrarily small neighborhood of an arbitrary dense open subset of Y.

If Y is a non-empty irreducible subvariety of an affine variety X prove that the stalk $\mathcal{O}_{X,Y}$ of $\mathcal{O}_X$ at Y is a K-algebra isomorphic to the localization $A(X)_{I(Y)}$ (hence giving a geometric meaning to this algebraic localization).