Exercise Sheet 4

As in class let $\mathbb{P}^1$ be the prevariety obtained by gluing two copies of the affine line $\mathbb{A}^1$ along the isomorphism $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$, $x \to \frac{1}{x}$. By the inclusion of one of the copies, we consider $\mathbb{A}^1$ as an open subprevariety of $\mathbb{P}^1$.

Exercise 1. Which of the following ringed spaces are isomorphic over $\mathbb{C}$?

a) $\mathbb{A}^1$

b) $V(x_1^2 + x_2^2) \subseteq \mathbb{A}^2$

c) $V(x_2 - x_1^2, x_3 - x_1^3) \setminus \{0\} \subseteq \mathbb{A}^3$

d) $V(x_2^2 - x_1^2 x_3 - x_1^3) \setminus \{0\} \subseteq \mathbb{A}_3$

e) $V(x_1 x_2) \subseteq \mathbb{A}^2$

f) $\mathbb{A}^1 \setminus \{1\}$

Exercise 2. Let $f : X \to Y$ be a morphism of affine varieties and $f^* : A(Y) \to A(X)$ the corresponding homomorphism of the coordinate rings. Are the following statements true or false?

a) f is surjective if and only if f^* is injective.

b) f is injective if and only if f^* is surjective.

c) If $f : \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism then f is affine linear i.e. of the form $f(x) = ax+b$ for some $a, b \in K$.

d) If $f : \mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism then f is affine linear i.e. it is of the form $f(x) = Ax + b$ for some $A \in \mathrm{Mat}(2 \times 2, K)$ and $b \in K^2$.

Exercise 3. Prove the following statements:

a) Every morphism $\mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^1 \to \mathbb{P}^1$.

b) Not every morphism $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^2 \to \mathbb{P}^1$.

c) Every morphism $\mathbb{P}^1 \to \mathbb{A}^1$ is constant.

Exercise 4. If X and Y are affine varieties we have seen that there is a bijection

$$\{\text{morphisms } X \to Y\} \overset{1:1}{\longleftrightarrow} \{\text{K-algebra homomorphisms } \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)\}, \quad f \mapsto f^*.$$

a) Does this statement still hold if X is an arbitrary prevariety (but Y is still affine)?

b) Does this statement still hold if Y is an arbitrary prevariety (but X is still affine)?