Exercise Sheet 5

Exercise 1.

a) Compute explicit generators for the ideal $I(a) \subseteq K[x_0, \dots, x_n]$ of an arbitrary point $a \in \mathbb{P}^n$.

b) Let $X = V(x_1^2 - x_2^2 - 1, x_3 - x_1) \subset \mathbb{A}^3_{\mathbb{C}}$. What are the points at infinity of the projective closure $\overline{X} \subset \mathbb{P}^3_{\mathbb{C}}$, i.e., the points in $\overline{X} \backslash X$?

Exercise 2. A line in $\mathbb{P}^n$ is a linear subspace of dimension 1.

a) Let $L_1, L_2 \subset \mathbb{P}^3$ be two disjoint lines and let $a \in \mathbb{P}^3 \backslash (L_1 \cup L_2)$. Show that there is a unique line $L \subset \mathbb{P}^3$ through a that intersects both L_1 and L_2.
Hint: Think about the corresponding cones in $\mathbb{A}^4$.

b) Is the corresponding statement for lines and points in $\mathbb{A}^3$ true as well?

Exercise 3.

a) Prove that a graded ring R is an integral domain if and only if for all homogeneous elements $f, g \in R$ with $fg = 0$ we have $f = 0$ or $g = 0$.

b) Show that a projective variety X is irreducible if and only if its homogeneous coordinate ring $S(X)$ is an integral domain.

Exercise 4. In this exercise we want to show that an intersection of projective varieties is never empty unless one would expect it to be empty for dimensional reasons — so, e.g., the phenomenon of parallel non-intersecting lines in the plane does not occur in projective space.

a) Let $X, Y \subset \mathbb{A}^n$ be pure-dimensional affine varieties. Show that every irreducible component of $X \cap Y$ has dimension at least $\dim X + \dim Y - n$.
Hint: Use diagonals.

b) Now let $X \subseteq \mathbb{P}^n$ be a non-empty projective variety. Prove that the dimension of the cone $C(X) \subset \mathbb{A}^{n+1}$ is $\dim X + 1$.

c) Let $X, Y \subset \mathbb{P}^n$ be projective varieties with $\dim X + \dim Y \geq n$. Show that $X \cap Y \neq \emptyset$.