Exercise Sheet 6

Exercise 1. Let $m, n \in \mathbb{N}_{>0}$. Prove:

a) If $f : \mathbb{P}^n \to \mathbb{P}^m$ is a morphism and $X \subset \mathbb{P}^m$ a hypersurface then every irreducible component of $f^{-1}(X)$ has dimension at least $n-1$.
Hint: Use that locally on an affine cover of $\mathbb{P}^m$, the hypersurface X is cut out by a single equation.

b) Show that any morphism $f : \mathbb{P}^n \to \mathbb{A}^m$ is constant.

c) If $n > m$ then every morphism $f : \mathbb{P}^n \to \mathbb{P}^m$ is constant.
Hint: Consider the preimages of the hyperplanes $V(x_i) \subseteq \mathbb{P}^m$ under f.

d) $\mathbb{P}^n \times \mathbb{P}^m$ is not isomorphic to $\mathbb{P}^{n+m}$.

Exercise 2. Let us say that $n+2$ points in $\mathbb{P}^n$ are in general position if for any $n+1$ of them their representatives in K^{n+1} are linearly independent.
Now let $a_1, \ldots, a_{n+2}$ and $b_1, \ldots, b_{n+2}$ be two sets of points in $\mathbb{P}^n$ in general position.

a) Show that the collection $A_1 = e_0 = (1 : 0 : \ldots : 0)$, $A_2 = e_1 = (0 : 1 : 0 : \ldots : 0)$, $\ldots$, $A_{n+1} = e_n = (0 : \ldots : 0 : 1)$, $A_{n+1} = (1 : 1 : \ldots : 1)$ is in general position.

b) Show that there is an isomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$ with $f(a_i) = b_i$ for all $i = 1, \ldots, n+2$.

Exercise 3. Show by example that the homogeneous coordinate ring of a projective variety is not invariant under isomorphisms i.e., that there are isomorphic projective varieties X, Y such that the graded K-algebras $S(X)$ and $S(Y)$ are not isomorphic.

Exercise 4. A conic over a field of characteristic not equal to 2 is an irreducible curve in $\mathbb{P}^2$ of degree 2.

a) Using the coefficients of quadratic polynomials show that the set of all conics can be identified with an open subset U of $\mathbb{P}^5$. (One says that U is a moduli space for conics.)

b) Given a point $p \in \mathbb{P}^2$ show that the subset of U consisting of all conics passing through p is the zero locus of a linear equation in the homogeneous coordinates of $U \subset \mathbb{P}^5$.

c) Given 5 points in $\mathbb{P}^2$, no three of which lie on a line, show that there is a unique conic passing through all these points.