Exercise Sheet 7

Exercise 1. Let $X \subseteq \mathbb{P}^3$ be the degree-3 Veronese embedding of $\mathbb{P}^1$, i.e., the image of the morphism

$$\mathbb{P}^1 \to \mathbb{P}^3, \quad (x_0 : x_1) \to (y_0 : y_1 : y_2 : y_3) = (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3).$$

Moreover, let $a = (0 : 0 : 1 : 0) \in \mathbb{P}^3$ and $L = V(y_2) \subseteq \mathbb{P}^3$ and let f be the projection from a to L.

a) Determine an equation of the curve $f(X)$ in $L \cong \mathbb{P}^2$.

b) Is $f : X \to f(X)$ an isomorphism onto its image?

Exercise 2.

a) For any $n, d \in \mathbb{N}_{>0}$, find explicit equations describing the image of the degree-d Veronese embedding

$$F : \mathbb{P}^n \to \mathbb{P}^N, (x_i)_{i=0,\dots,n} \mapsto (z_\alpha)_{\alpha \in \mathbb{N}^{n+1} : \sum \alpha_i = d} = \left(x^\alpha = \prod_{i=0}^n x_i^{\alpha_i} \right)_\alpha$$

of $\mathbb{P}^n$ in $\mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$.

b) Prove that every projective variety is isomorphic to the zero locus of quadratic polynomials in a projective space.

Exercise 3. We denote the Plücker coordinates of the Grassmannian $G(2, 4)$ in $\mathbb{P}^5$ by x_{ij} for $1 \leq i < j \leq 4$.

a) Show that $G(2, 4) = V(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})$.

b) Let $L \subseteq \mathbb{P}^3$ be an arbitrary line. Show that the set of lines in $\mathbb{P}^3$ that intersect L, considered as a subset of $G(2, 4) \subseteq \mathbb{P}^5$, is the zero locus of a homogeneous linear polynomial.

How many lines in $\mathbb{P}^3$ would you expect to intersect four general given lines?

Exercise 4. Show that the following sets are projective varieties:

a) the incidence correspondence

$$\{(L, a) \in G(k, n) \times \mathbb{P}^{n-1} : L \subseteq \mathbb{P}^{n-1} \text{ a } (k-1)\text{-dimensional linear subspace and } a \in L\};$$

b) the join of two disjoint varieties $X, Y \subseteq \mathbb{P}^n$, i.e., the union of all lines in $\mathbb{P}^n$ intersecting both X and Y.