# Exercise Sheet 8

**Exercise 1.** Let $\widetilde{\mathbb{A}}^3$ be the blow-up of $\mathbb{A}^3$ at the line $L = V(x_1, x_2) \cong \mathbb{A}^1$. Show that its exceptional set is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$. When do the strict transforms of two lines in $\mathbb{A}^3$ through $L$ intersect in the blow-up? What is therefore the geometric meaning of the points in the exceptional set (analogously to the blow-up of a point, in which case the points of the exceptional set correspond to the directions through the blown-up point)?

**Exercise 2.** Show that any irreducible quadric hypersurface $Q \subseteq \mathbb{P}^n$ over a field of characteristic not equal to 2 is birational to $\mathbb{P}^{n-1}$. Can you give an example of some $Q$ which is not isomorphic to $\mathbb{P}^{n-1}$?

**Exercise 3.** Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $Y_1, Y_2 \subseteq X$ be irreducible, closed subsets, none contained in the other. Moreover, let $\widetilde{X}$ be the blow-up of $X$ at the ideal $I(Y_1) + I(Y_2)$. Show that the strict transforms of $Y_1$ and $Y_2$ in $\widetilde{X}$ are disjoint.

**Exercise 4.** Let $J \trianglelefteq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal, and assume that the corresponding affine variety $X = V(J) \subseteq \mathbb{A}^n$ contains the origin. Consider the blow-up $\widetilde{X} \subseteq \widetilde{\mathbb{A}}^n \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ at $x_1, \ldots, x_n$, and denote the homogeneous coordinates of $\mathbb{P}^{n-1}$ by $y_1, \ldots, y_n$.

*a)* We know already that $\widetilde{\mathbb{A}}^n$ can be covered by affine spaces, with one coordinate patch being

$$i : U = \mathbb{A}^n \to \widetilde{\mathbb{A}}^n \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1},$$
$$(x_1, y_2, \ldots, y_n) \mapsto ((x_1, x_1 y_2, \ldots, x_1 y_n), (1 : y_2 : \cdots : y_n)).$$

Prove that on this coordinate patch the blow-up $\widetilde{X}$ is given as the zero locus of the polynomials

$$\frac{f(x_1, x_1 y_2, \ldots, x_1 y_n)}{x_1^{\text{min deg } f}}$$

for all non-zero $f \in J$, where min deg $f$ denotes the smallest degree of a monomial in $f$.

*Hint:* You can use without proof the following variant of [Gathmann, Exercise 2.23]:

For $I, J \trianglelefteq K[x_1, \ldots, x_n]$ one has $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$ where
$$(I : J^\infty) = \{f \in K[x_1, \ldots, x_n] : \exists m \in \mathbb{N}, g \in J^m \text{ with } fg \in I\}.$$

*b)* Show that the exceptional set of the blow-up $\widetilde{X}$ is

$$V_p\left(f^{\text{in}}(y) : f \in J\right) \subseteq \mathbb{P}^{n-1} \cong \{0\} \times \mathbb{P}^{n-1},$$

where $f^{\text{in}}$ is the initial term of $f$, i.e. the sum of all monomials in $f$ of smallest degree. Consequently, the tangent cone of $X$ at the origin is

$$C_0 X = V_a(f^{\text{in}} : f \in J) \subseteq \mathbb{A}^n.$$