Exercise Sheet 1

Exercise 1. Consider the ideal $J = \langle x_i - x_1 x_{i-1} : i = 2, \dots, n \rangle \trianglelefteq K[x_1, \dots, x_n]$ and let $X = V(J) \subseteq \mathbb{A}^n$ be its vanishing locus.

(a) Show that the projection

$$\pi : X \to \mathbb{A}^1, (x_1, \dots, x_n) \mapsto x_1$$

is a bijection. Calculate the inverse map $\pi^{-1} : \mathbb{A}^1 \to X$ parameterizing X.

(b) Show that J is a prime ideal.
Hint: Calculate the quotient $K[x_1, \dots, x_n]/J$.

(c) Conclude that $J = I(X)$ and compute the coordinate ring $A(X)$.

Solution.

(a) For a fixed $x_1 \in \mathbb{A}^1$, let $x = (x_1, x_2, \dots, x_n) \in \pi^{-1}(\{x_1\})$ be a point in the preimage. Then the vanishing of the generators of J at x implies $x_2 - x_1 \cdot x_1 = 0$ and thus $x_2 = x_1^2$. Furthermore, we have $x_3 - x_1 x_2 = 0$ so $x_3 = x_1^3$. Inductively, one shows that $x = (x_1, x_1^2, x_1^3, \dots, x_1^n)$ is the unique point on which all generators of J vanish. The associated inverse map to π is given by

$$x_1 \mapsto (x_1, x_1^2, x_1^3, \dots, x_1^n) .$$

(b) Similar to part (a), we can inductively simplify the quotient ring:

$$\begin{aligned} &K[x_1, x_2, x_3, \dots, x_n]/\langle x_2 - x_1^2, x_3 - x_1 x_2, \dots \rangle \\ \cong &K[x_1, x_3, \dots, x_n]/\langle x_3 - x_1 x_1^2, \dots \rangle \dots \\ \cong &K[x_1, x_n]/\langle x_n - x_1^n \rangle \cong K[x_1] . \end{aligned}$$

Here we used the general result: for R a ring and $f \in R$ the map $R \to R[x]/(x - f)$ is an isomorphism. Since $K[x_1]$ is a domain, we conclude that J is a prime ideal.

(c) By the Nullstellensatz, we have $I(X) = I(V(J)) = \sqrt{J} = J$ where the last equality follows since any prime ideal is radical. By a result from the lecture, we have $A(X) = K[x_1, \dots, x_n]/I(X) \cong K[x_1]$ by the calculation in part (b).

Exercise 2. Determine the radical of the ideal $J = \langle x_1^3 - x_2^6, x_1 x_2 - x_2^3 \rangle \subseteq \mathbb{C}[x_1, x_2]$.
Hint: The Nullstellensatz might be useful here.

Solution. For $X = V(J)$ we have $\sqrt{J} = I(X)$ by the Nullstellensatz. Thus we first calculate X. For a point $x \in \mathbb{C}^2$ we have $x \in X$ implies $x_1 x_2 - x_2^3$ and so $x_2 = 0$ or

$x_1 = x_2^2$. For $x_2 = 0$ the equation $x_1^3 - x_2^6$ shows $x_1 = 0$ leading to the solution $x = (0, 0)$. On the other hand, for $x_1 = x_2^2$ the equation $x_1^3 - x_2^6$ is automatically satisfied, leading to the solution $x \in V(x_1 - x_2^2)$, which also contains $(0, 0)$. Thus $X = V(x_1 - x_2^2)$. We conclude

$$\sqrt{J} = I(X) = I(V(\langle x_1 - x_2^2 \rangle)) = \langle x_1 - x_2^2 \rangle \,,$$

because $\langle x_1 - x_2^2 \rangle$ is prime, hence radical. Indeed, the ring $K[x_1, x_2]/\langle x_1 - x_2^2 \rangle \cong K[x_2]$ is a domain, which proves that the ideal is prime.

Exercise 3. Let $X \subset \mathbb{A}^n$ be an affine variety. Show that the coordinate ring $A(X)$ is a field if and only if X is a single point.

Solution. We can either prove the exercise using a consequence of the Nullstellensatz, or give a hands-on argument:
Fany argument:

The ring $A(X) = K[x_1, \ldots, x_n]/I(X)$ is a field if and only if $I(X)$ is a maximal ideal. In the lecture we saw that maximal ideals correspond to points of $\mathbb{A}^n$, so these are the only affine varieties with $A(X)$ a field.
Argument "by hand":

If X is a point $X = \{a\}$, then any polynomial function f on X is uniquely determined by $f(a) \in K$, and conversely those (constant) functions are polynomial. Thus $A(X) = K$ is a field.

Conversely, assume X is not a point. For $X = \emptyset$ we have $A(\emptyset) = \{0\}$ the zero ring, which is not a field. On the other hand, if X has at least two distinct points a_1, a_2, then there is a linear function f with $f(a_1) = 0, f(a_2) = 1$. Clearly $f \neq 0$ in $A(X)$, but f cannot have a multiplicative inverse g since $(f \cdot g)(a_1) = 0$ for any possible $g \in A(X)$.

Exercise 4. Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes.

(a) Compute generators for the ideal $I(X)$.

(b) Show that $I(X)$ cannot be generated by fewer than three elements.

Solution.

(a) We have $I(X) = \langle x_1x_2, x_1x_3, x_2x_3 \rangle =: J$. Indeed, if $x = (x_1, x_2, x_3) \in V(J)$ and $x_1 = 0$ then $x_2x_3 = 0$, so $x_2 = 0$ or $x_3 = 0$. If $x_1 \neq 0$ then the equations $x_1x_2 = x_1x_3 = 0$ imply $x_2 = x_3 = 0$. Going through the cases, we see that $V(J)$ is precisely the union X of the coordinate axes. Thus by the Nullstellensatz $I(X) = I(V(J)) = \sqrt{J}$. Hence it suffices to show that J is radical, which we do by proving that $R = K[x_1, x_2, x_3]/J$ is reduced.

For doing this, note that any element $f = f(x_1, x_2, x_3)$ in the quotient ring R has a (unique) representative by a polynomial that has no cross-terms (i.e. no monomial is divisible by two or more of the variables). Thus we can write any element of R as

$$f = c + x_1 f_1(x_1) + x_2 f_2(x_2) + x_3 f(x_3) \in R$$

with $f_1, f_2, f_3 \in K[x]$. Assume that $f^m = 0$ for some m, then expanding this out, we see that the unique representative of f^m has constant term c^m. The condition $f^m = 0 \in R$ then implies $c = 0$. Looking at the other terms, we see

$$f^m = x_1 \cdot (x_1^{m-1} f_1(x_1)^m) + x_2 \cdot (x_2^{m-1} f_2(x_2)^m) + x_3 \cdot (x_3^{m-1} f_3(x_1)^m) + J \in R.$$

This implies $x_i^{m-1} f_i(x_i) = 0$ for $i = 1, 2, 3$ and thus $f_i = 0$. Thus 0 is the only nilpotent element in R, making it a reduced ring.

(b) Assume $I(X) = \langle f_1, f_2 \rangle$ was generated by two polynomials f_i. Since the generators of $I(X)$ are of pure degree 2, the two elements f_1, f_2 satisfy that any monomial contained in them has degree at least 2 as well. Let

$$V = K[x_1, x_2, x_3]_2 = \mathrm{Span}_K\{x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\}$$

be the span of monomials of degree exactly 2, forming a K-vector subspace of $K[x_1, x_2, x_3]$. We have $V \cap I(X) = \mathrm{Span}_K\{x_1x_2, x_1x_3, x_2x_3\}$ has dimension 3. On the other hand, the degree 2 part of an element $g_1f_1 + g_2f_2 \in \langle f_1, f_2 \rangle$ is obtained as the sum of the constant parts of the g_i multiplied by the homogeneous degree 2 parts of f_i. Thus $V \cap \langle f_1, f_2 \rangle$ is spanned by these homogeneous degree 2 parts of f_1, f_2, and thus has dimension at most two. This is a contradiction to the equation above.