Exercise Sheet 10

Exercise 1. As in Sheet 9, Exercise 3, let $U \subseteq \mathbb{P}^{\binom{4+5}{4}-1} = \mathbb{P}^{125}$ be the set of all smooth (3-dimensional) hypersurfaces of degree 5 in $\mathbb{P}^4$.

a) Using the Jacobi criterion, show that the incidence correspondence

$$M := \{(X, L) \in U \times G(2,5) : L \text{ is a line contained in } X\}$$

is smooth of dimension 125, i.e., of the same dimension as U.
Hint: Reduce to showing smoothness when $L = \mathrm{Lin}(e_1, e_2)$. Then for $X = V_p(f_c)$ with

$$f_c = c_0x_0^5 + c_1x_0^4x_1 + \ldots + c_5x_1^5 + c_6x_0^4x_2 + \ldots$$

apply the Jacobi criterion for the partial derivatives along the variables $c_0, \ldots, c_5$.

b) Although (a) suggests that a smooth hypersurface of degree 5 in $\mathbb{P}^4$ contains only finitely many lines, show that the Fermat hypersurface $V(x_0^5 + \cdots + x_4^5) \subset \mathbb{P}^4$ contains infinitely many lines.
Hint: Consider lines of the form $L = \{(a_0s : a_1s : a_2t : a_3t : a_4t) : (s : t) \in \mathbb{P}^1\}$ for suitable $a_0, \ldots, a_4 \in \mathbb{C}$.)

Solution.

a) As in the proof of [Gathmann, Lemma 11.4] we can use the action of $\mathrm{PGL}(5, K)$ on $\mathbb{P}^4$ to ensure that $L = \mathrm{Lin}(e_1, e_2)$, so that we obtain affine coordinates

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 & a_4 \\ 0 & 1 & b_2 & b_3 & b_4 \end{pmatrix}$$

on $G(2,5)$ in a neighborhood of L. For coordinates $(c, a, b) = ((c_\alpha)_\alpha, a_2, \ldots, a_4, b_2, \ldots, b_4)$ on $U \times G(2,5)$ we have

$$(c, a, b) \in M \iff f_c(s(1, 0, a_2, a_3, a_4) + t(0, 1, b_2, b_3, b_4)) = 0 \text{ for all } s, t\,.$$

The polynomial on the right is homogeneous of degree 5 in s, t, so as in the proof of [Gathmann, Lemma 11.4] this is equivalent to the vanishing of the six coefficient functions $F_i(c, a, b)$ of s^it^{5-i} for $i = 0, \ldots, 5$. The Jacobi matrix J is now a $6 \times (125 + 6)$-matrix, and we want to show that it is invertible by choosing six of its columns and calculating that the resulting 6×6-matrix has full rank. In the previous proof in the script, the last 6 rows (with partial derivatives along the components of a, b) were chosen. It follows from the example in part b) that this can no longer work here, since M is no longer locally the graph of a function $U \to G(2,5)$ (since there can be $X \in U$ having infinitely many points $(X, L) \in M$ lying over it).

Instead, as suggested by the hint we want to take the partial derivatives along the components $c_0, \ldots, c_5$ of c.[1] Expanding the term $f_c(s(1,0,a_2,a_3,a_4)+t(0,1,b_2,b_3,b_4))$ above, we see that it equals

$$c_0 s^5 + c_1 s^4 t + \ldots + c_5 t^5 + R(c_6, \ldots, c_{125}, a, b, s, t)\,,$$

for some polynomial R. Extracting the coefficient F_i of $s^i t^{5-i}$ we see $F_i = c_i + R_i(c_6, \ldots, c_{125}, a, b)$ for suitable polynomials R_i. Calculating the partial derivatives along $c_0, \ldots, c_5$ we find

$$\frac{\partial F_i}{\partial c_j} = \delta^i_j\,,$$

where δ^i_j is the Kronecker delta. Thus in fact the first 6 columns of the matrix J are just the identity matrix, and thus invertible.

b) Plugging in the suggested lines into the equation of the Fermat hypersurface, we find the condition

$$(a_0 s)^5 + (a_1 s)^5 + (a_2 t)^5 + (a_3 t)^5 + (a_4 t)^5 = 0 \in \mathbb{C}[s,t]$$
$$\iff a_0^5 + a_1^5 = 0 \text{ and } a_2^5 + a_3^5 + a_4^5 = 0$$

These two equations cut out a surface of points $a \in \mathbb{P}^4$ satisfying this, and thus there are infinitely many lines.

Exercise 2. Find an example of the following, or prove that it does not exist:

a) an irreducible affine scheme $\operatorname{Spec} R$ such that R is not an integral domain;

b) a point of $\operatorname{Spec}(\mathbb{R}[x_1,x_2]/\langle x_1^2 + x_2^2 + 1\rangle)$ with residue field $\mathbb{R}$;

c) two affine schemes $\operatorname{Spec} R$ and $\operatorname{Spec} S$ with $R \subseteq S$ and $\dim(\operatorname{Spec} R) > \dim(\operatorname{Spec} S)$;

d) an affine scheme of dimension 1 with exactly two points.

Solution.

a) Let $R = K[x]/\langle x^2\rangle$, then this is a local ring for which the nilradical $\sqrt{\langle 0\rangle} = \langle x\rangle$ is also maximal, making it the only prime ideal of R. Thus $\operatorname{Spec}(R) = \{\langle x\rangle\}$ is a one-point space and thus irreducible. On the other hand, the ring R is not an integral domain since $x \neq 0$ but $x \cdot x = 0 \in R$.

b) Let $R = \mathbb{R}[x_1,x_2]/\langle x_1^2 + x_2^2 + 1\rangle$ and assume we have $p \subseteq R$ prime with residue field $\mathbb{R}$, i.e. $\operatorname{Frac}(R/p) = \mathbb{R}$ so that we obtain a ring homomorphism $\varphi : R \to \mathbb{R}$. Let $a_1 = \varphi(x_1), a_2 = \varphi(x_2) \in \mathbb{R}$ be the images of the two coordinate functions. Then

$$0 = \varphi(x_1^2 + x_2^2 + 1) = \varphi(x_1)^2 + \varphi(x_2)^2 + 1 = a_1^2 + a_2^2 + 1 \in \mathbb{R}\,.$$

But such elements $a_1, a_2 \in \mathbb{R}$ cannot exist since $a_1^2 + a_2^2 \geq 0$. Thus no such point $p \in \operatorname{Spec}(R)$ can exist.

[1] Here we work in some chart on the projective space $\mathbb{P}^{125}$ where one of the other coordinates c_j was set to 1.

c) Take $R = \mathbb{Z} \subseteq S = \mathbb{Q}$. Since $\mathbb{Q}$ is a field, it has unique prime ideal $\langle 0 \rangle$, so $\mathrm{Spec}(S) = \{\langle 0 \rangle\}$ is a one-element set of dimension 0. On the other hand, in $\mathbb{Z}$ we have a strict inclusion of prime ideals $\langle 0 \rangle \subsetneq \langle 2 \rangle$ leading to a chain $V(2) \subsetneq V(0) = \mathrm{Spec}(R)$ so that $\dim \mathrm{Spec}(R) \geq 1$ (and in fact $\dim \mathrm{Spec}(R) = 1$).

d) Take $R = \mathbb{Z}_{\langle 2 \rangle}$ be the localization of the integers at the prime ideal $\langle 2 \rangle$ (which is different from the localization $\mathbb{Z}_2$ at the element 2!). From commutative algebra we know that the prime ideals of R are in correspondence to the prime ideals of $\mathbb{Z}$ contained in $\langle 2 \rangle$, which are precisely

$$\mathrm{Spec}(R) = \{\langle 2 \rangle, \langle 0 \rangle\}\,.$$

Thus $\mathrm{Spec}(R)$ has precisely two points, and the inclusion $\{\langle 2 \rangle\} \subseteq \mathrm{Spec}(R)$ is the unique maximal chain of irreducible closed subsets, and since it has length 1 we know $\dim \mathrm{Spec}(R) = 1$.

Exercise 3.

a) Let $R = A(X)$ be the coordinate ring of an affine variety X over an algebraically closed field. Show that the set of all closed points is dense in $\mathrm{Spec}\, R$ (which means by definition that every non-empty open subset of $\mathrm{Spec}\, R$ contains a closed point).

b) In contrast to (a), however, show by example that on a general affine scheme the set of all closed points need not be dense.

Solution.

a) Let $U \subseteq \mathrm{Spec}(R)$ be a non-empty open subset with closed complement $V(J)$ for $J \subseteq R$ an ideal. Assume that all closed points of $\mathrm{Spec}(R)$ are contained in $V(J)$. We have seen (by the Nullstellensatz) that the closed points, corresponding to the maximal ideals of R, are exactly given by $m_a = I(\{a\})$ for $a \in X$. The containment $m_a \in V(J)$ is equivalent to $J \subseteq m_a$, meaning that all functions $f \in J$ vanish at a. Thus for all $f \in J$ we have that f vanishes at any point $a \in X$ and thus $f \in I(X)$. By the Nullstellensatz we have $I(X) = I(V(\langle 0 \rangle)) = \sqrt{\langle 0 \rangle} = \langle 0 \rangle$, using that R is reduced. Thus $J = \langle 0 \rangle$, giving a contradiction since $U = \mathrm{Spec}(R) \setminus V(0) = \emptyset$.

b) For the example of $R = \mathbb{Z}_{\langle 2 \rangle}$ above we have that $\langle 2 \rangle$ is the unique closed point, but its closure is not everything (it's just $\{\langle 2 \rangle\}$ again).