Exercise Sheet 11

Exercise 1. Let R be a ring. Prove that the affine scheme $\operatorname{Spec} R$ is disconnected if and only if $R \cong S \times T$ for two non-zero rings S and T.

Solution. First, assume that $X = \operatorname{Spec}(R)$ is disconnected, decomposing as $X = X_1 \sqcup X_2$ for two open disjoint subsets $X_1, X_2 \subseteq X$. Then we claim that

$$R = \mathcal{O}_X(X) = \underbrace{\mathcal{O}_X(X_1)}_{=:S} \times \underbrace{\mathcal{O}_X(X_2)}_{=:T}$$

Indeed, a regular function $\varphi = (\varphi_p)_{p \in X}$ on X is uniquely determined by its restrictions $\varphi_1 = (\varphi_p)_{p \in X_1} \in S$ and $\varphi_2 = (\varphi_p)_{p \in X_2} \in T$, and conversely any such pair gives rise to a function φ as above. The reason is that the condition on regular functions is local, and thus can be checked on the open cover $X = X_1 \sqcup X_2$, where it restricts to the corresponding conditions on regular functions for X_1 and X_2.

Conversely, assume that $R \cong S \times T$, then we claim that $\operatorname{Spec} R \cong \operatorname{Spec} S \sqcup \operatorname{Spec} T$. Since S, T are nonzero rings, the ideal $\langle 0 \rangle \subseteq S$ is a proper ideal and thus contained in some maximal (and hence prime) ideal of S. Thus S (and T) have non-empty spectrum, finishing the proof.

To show the claim, consider the two elements $f = (1,0), g = (0,1) \in S \times T$. Then we claim that we have isomorphisms

$$S \xrightarrow{\sim} (S \times T)_f, T \xrightarrow{\sim} (S \times T)_g \text{ and } \{0\} \xrightarrow{\sim} (S \times T)_{fg},$$

where e.g. the first map is given by $s \mapsto (s,0)$ with inverse given by $(s,t)/f^m \mapsto s$. To check that these are isomorphic, one just observes that $(s,t)/f^m = (s,0) \in (S \times T)_f$ since

$$f \cdot ((s,t) - (s,0) \cdot f^m) = (1,0) \cdot ((s,t) - (s,0) \cdot (1^m, 0^m)) = (1,0) \cdot (0,t) = (0,0) \in S \times T.$$

The other proofs work similar. This shows that the two open sets $D(f), D(g) \subseteq \operatorname{Spec}(R)$ are isomorphic to $\operatorname{Spec}((S \times T)_f) = \operatorname{Spec}(S)$ and $\operatorname{Spec}(T)$ and that their intersection $D(fg) = D(0)$ is empty. This proves

$$\operatorname{Spec}(R) = D(f) \sqcup D(g) \cong \operatorname{Spec}(S) \times \operatorname{Spec}(T).$$

Exercise 2. For $n \in \mathbb{N}_{>0}$, an n-fold point over an algebraically closed field K is an affine scheme $\operatorname{Spec} R$ that contains only one point, and such that R is a K-algebra of vector space dimension n over K.

a) Show that every single point over K is isomorphic to $\operatorname{Spec} K$.

b) Show that every double point over K is isomorphic to $\operatorname{Spec} K[x]/\langle x^2 \rangle$.

c) Is part b) correct without the assumption that K is algebraically closed?

d) Find two non-isomorphic triple points over an algebraically closed field K. Here we mean: there is no isomorphism $\operatorname{Spec} R_1 \xrightarrow{\sim} \operatorname{Spec} R_2$ coming from a K-algebra homomorphism $R_2 \to R_1$. Can you describe them geometrically?

Solution. By definition, a K-algebra R is a ring R together with a ring homomorphism $\varphi : K \to R$. The kernel of φ is an ideal in K, and so it must be $\{0\}$ or all of K. Since ring homomorphisms send 1 to 1, the second possibility is excluded (for $R \neq \{0\}$), and so φ is injective. Using it to define a K-vector space structure on R, it automatically becomes a K-linear map.

a) By the preparatory comments, we have that R is a one-dimensional vector space, containing $\varphi(K) \cong K$ and thus φ in fact defines an isomorphism $K \cong R$.

b) We have $1 \neq 0 \in R$ and so we can enrich it to a basis $1, a$ of R as a K-vector space. Since the product in R is K-bilinear, it is uniquely determined by the products of the basis elements with themselves. The products $1 \cdot 1 = 1$ and $1 \cdot a = a \cdot 1 = a$ are automatic. Let $r, s \in K$ be such that $a \cdot a = r \cdot 1 + s \cdot a$, which are uniquely determined since $1, a$ are a K-vector space basis of R. Then clearly

$$R \cong K[a]/\langle a^2 - (r + sa)\rangle \,.$$

However, since K is algebraically closed, the polynomial $f = x^2 - sx - r$ decomposes into linear factors $f = (x - u_1) \cdot (x - u_2)$ for $u_1, u_2 \in K$. If $u_1 \neq u_2$, then the ring R above has two different maximal ideals (spanned by $a - u_1$ and $a - u_2$), giving a contradiction to $\operatorname{Spec} R$ having only one point. Thus $u_1 = u_2 = u \in K$ and $a^2 - sa - r = (a - u)^2$. Setting $x = a - u$ we have $R \cong K[x]/\langle x^2\rangle$.

c) No: take $K = \mathbb{R}$ and $R = \mathbb{C}$, then R is a K-algebra with $\dim_K R = 2$ but $\mathbb{C}$ is reduced, whereas $\mathbb{R}[x]/\langle x^2\rangle$ is non-reduced.

d) We can take $R_1 = K[x]/\langle x^3\rangle$ and $R_2 = K[x, y]/\langle x^2, xy, y^2\rangle$. They have unique prime ideals $p_1 = \langle x\rangle$ and $p_2 = \langle x, y\rangle$. The Zariski cotangent spaces are $p_1/p_1^2 \cong \langle x\rangle/\langle x^2\rangle \cong K$ and $\langle x, y\rangle/\langle x^2, xy, y^2\rangle \cong K^2$ and thus of different dimensions over K. Alternatively, we can just see that $p_1^2 \neq 0$ but $p_2^2 = 0$. This shows that R_1, R_2 cannot be isomorphic as K-algebras.
We can see $\operatorname{Spec} R_1$ as a fat point stretched along one dimension, whereas $\operatorname{Spec} R_2$ is stretched equally in two dimensions.

Exercise 3.

a) Let K be a field. Show that $\operatorname{Spec} K[x]/\langle x^3 - x^2\rangle \cong \operatorname{Spec} K[x]/\langle x^2\rangle \sqcup \operatorname{Spec} K$.

b) For $R = K[x, y]$ calculate the scheme-theoretic intersection $X_1 \cap X_2$ of the two affine subschemes

$$X_1 = \operatorname{Spec} R/\langle x^2 + y^2 - 1\rangle \to \operatorname{Spec} R \text{ and } X_2 = \operatorname{Spec} R/\langle y - x^2 + 1\rangle \to \operatorname{Spec} R\,.$$

How many connected components does $X_1 \cap X_2$ have?

Solution.

a) The two ideals $I = \langle x^2 \rangle$ and $J = \langle x-1 \rangle$ in $K[x]$ are coprime since

$$1 = -(x-1)\cdot(x+1) + x^2 \in I + J\,.$$

Also since $K[x]$ is a unique factorization domain, their intersection is given by $I \cap J = \langle x^2 \cdot (x-1)\rangle$. Thus by the Chinese remainder theorem we have

$$K[x]/\langle x^3 - x^2\rangle \cong K[x]/\langle x^2\rangle \times \underbrace{K[x]/\langle x-1\rangle}_{\cong K}\,.$$

Using the result from Exercise 1 (that products of rings give disjoint unions of affine schemes) we conclude the claim.

b) By definition, the scheme-theoretic intersection $X_1 \cap X_2$ is cut out by the sum of the ideals, and so given by

$$\langle x^2 + y^2 - 1, y - x^2 + 1\rangle = \langle \underbrace{x^2 + (x^2-1)^2 - 1}_{=x^4 - x^2 = x^2(x-1)(x+1)}, y - x^2 + 1\rangle$$

Using the second generator to eliminate the variable $y = x^2 - 1$ from $K[x,y]$, we thus have

$$X_1 \cap X_2 \cong \operatorname{Spec} K[x]/\langle x^2(x-1)(x+1)\rangle\,.$$

By the Chinese remainder theorem again we have two cases:

- for $\mathrm{char}(K) \neq 2$ we have $1 \neq -1$ and

$$X_1 \cap X_2 \cong \operatorname{Spec} K[x]/\langle x^2\rangle \sqcup \underbrace{\operatorname{Spec} K[x]/\langle x-1\rangle}_{\cong \operatorname{Spec} K} \sqcup \underbrace{\operatorname{Spec} K[x]/\langle x+1\rangle}_{\cong \operatorname{Spec} K}$$

  is the disjoint union of one double point and two simple points.

- for $\mathrm{char}(K) = 2$ we have $1 = -1$ and

$$X_1 \cap X_2 \cong \operatorname{Spec} K[x]/\langle x^2\rangle \sqcup \underbrace{\operatorname{Spec} K[x]/\langle (x-1)^2\rangle}_{\cong \operatorname{Spec} K[y]/\langle y^2\rangle}$$

  is the disjoint union of two double points.