Exercise Sheet 13

Exercise 1.

a) Any morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves of modules on a scheme X determines induced $\mathcal{O}_{X,p}$-module homomorphisms

$$f_p : \mathcal{F}_p \to \mathcal{G}_p, [(U, \phi)] \mapsto [(U, f_U(\phi))],$$

on the stalks for all $p \in X$. Show that f is an isomorphism if and only if all f_p are isomorphisms.

b) Conclude that f is an isomorphism if and only if f is injective and surjective.

Solution.

a) If f is an isomorphism with inverse g, then we claim that g_p is an inverse of f_p, showing that f_p is an isomorphism for all $p \in X$. Indeed, since $f \circ g = \mathrm{id}_{\mathcal{F}}$ we have $f_p \circ g_p = \mathrm{id}_{\mathcal{F}_p}$ and similarly $g_p \circ f_p = \mathrm{id}_{\mathcal{G}_p}$. Therefore, f_p is an isomorphism for all $p \in X$.

Conversely, assume that f_p is an isomorphism for all $p \in X$. To show that f is an isomorphism, we need to construct an inverse morphism $g : \mathcal{G} \to \mathcal{F}$.

For each open set $U \subseteq X$ and each section $s \in \mathcal{G}(U)$, we can define $g_U(s)$ as follows. For each $p \in U$, $s_p \in \mathcal{G}_p$. Since f_p is an isomorphism, there exists a unique element $t_p = [(V_p, r_p)] \in \mathcal{F}_p$ such that $f_p(t_p) = [(V_p, f_{V_p}(r_p)] = s_p$. This last equality means that, up to shrinking V_p to a smaller neighborhood of p, we can in fact assume that $f_{V_p}(r_p) = s|_{V_p}$, by the definition of equality in $\mathcal{F}_p$.

We claim that the collection $\{(V_p, r_p)\}_{p \in U}$ of open subsets and sections of $\mathcal{G}$ is compatible on overlaps and thus glues to a unique section $t \in \mathcal{F}(U)$. Indeed, the equality on overlaps just follows since the stalks of $r_p|_{V_p \cap V_q}$ and $r_q|_{V_p \cap V_q}$ at all points $x \in V_p \cap V_q$ must coincide with $f_x^{-1}(s_x)$. Since the section of a sheaf is uniquely determined by all of its values at stalks, we have compatibility on overlaps and thus obtain $t \in \mathcal{F}(U)$.

Define $g : \mathcal{G} \to \mathcal{F}$ by sending $s \in \mathcal{G}(U)$ to $t \in \mathcal{F}(U)$ as constructed above. Then by construction $g_p = f_p^{-1}$ and using again the fact that sections of $\mathcal{F}, \mathcal{G}$ are determined by their values at stalks, one checks that g is the inverse morphism to f.

b) To conclude that f is an isomorphism if and only if f is injective and surjective, note the following:

- A morphism $f : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\ker(f) = 0$, i.e. if

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G}$$

is exact. Since exactness of sequences can be checked on stalks, f is injective if and only if for every $p \in X$, the induced map $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective.

- A morphism $f : \mathcal{F} \to \mathcal{G}$ is surjective if and only if the image sheaf of f is equal to $\mathcal{G}$, i.e. if

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \to 0$$

is exact. Again, since exactness of sequences can be checked on stalks, f is surjective if and only if for every $p \in X$, the induced map $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective.

Given part (a), f is an isomorphism if and only if f_p is an isomorphism for all $p \in X$. Since f_p being an isomorphism is equivalent to it being injective and surjective, we conclude that f is an isomorphism if and only if f is injective and surjective.

Exercise 2. Let $\mathcal{F}$ be a presheaf on a scheme X, and denote by $\theta : \mathcal{F} \to \mathcal{F}^{\text{sh}}$ its sheafification. Prove that any morphism $f' : \mathcal{F} \to \mathcal{G}$ to a sheaf $\mathcal{G}$ factors uniquely through θ, i.e., there is a unique morphism $f : \mathcal{F}^{\text{sh}} \to \mathcal{G}$ with $f' = f \circ \theta$.

Solution. Let $f'_p : \mathcal{F}_p \to \mathcal{G}_p$ be the map of stalks induced by f'. Let $U \subseteq X$ be an open set and $(s_p)_{p \in U}$ a section of $\mathcal{F}^{\text{sh}}$. By definition, each $p_0 \in U$ has an open neighborhood $U_{p_0} \subseteq U$ and $S_{p_0} \in \mathcal{F}(U_{p_0})$ such that $s_p = (S_{p_0})_p \in \mathcal{F}_p$ for all $p \in U_{p_0}$.

We define a morphism $f : \mathcal{F}^{\text{sh}}(U) \to \mathcal{G}(U)$ by setting $f((s_p)_{p \in U})$ to be the unique section in $\mathcal{G}(U)$ that restricts to $f'_{U_{p_0}}(S_{p_0})$ on each U_{p_0}. To see that this is well-defined, note that on overlaps $U_{p_0} \cap U_{q_0}$, the stalks of $f'_{U_{p_0}}(S_{p_0})$ and $f'_{U_{q_0}}(S_{q_0})$ at any point $p \in U_{p_0} \cap U_{q_0}$ agree (and are equal to $f'_p(s_p)$). Since $\mathcal{G}$ is a sheaf, this proves that the two sections agree, and that indeed we get a well-defined section $f((s_p)_{p \in U})$.

Now, we need to verify that $f \circ \theta = f'$. For any $U \subseteq X$ and $s \in \mathcal{F}(U)$, by the definition of θ, $\theta_U(s)$ is represented by the family $\{s\}$ on the open cover $\{U\}$. Thus, $f(\theta_U(s))$ should be the unique section in $\mathcal{G}(U)$ that restricts to $f'_U(s)$ on U, which is precisely $f'_U(s)$. Therefore, $f(\theta_U(s)) = f'_U(s)$, showing that $f \circ \theta = f'$.

To prove the uniqueness of f, assume there are two morphisms $f_1, f_2 : \mathcal{F}^{\text{sh}} \to \mathcal{G}$ such that $f' = f_1 \circ \theta = f_2 \circ \theta$. Since $\theta_p : \mathcal{F}'_p \to \mathcal{F}_p$ is an isomorphism, this equality forces $(f_1)_p = (f_2)_p$ for all $p \in X$. However, we already know that a morphism of sheaves is uniquely determined by its action on the stalks, so $f_1 = f_2$ as desired.

Exercise 3. Let X be a topological space and consider the presheaves

$$\mathcal{F}(U) = \{\varphi : U \to \mathbb{R} : \varphi \text{ continuous and bounded}\} \text{ for } U \subseteq X \text{ open}\,,$$
$$\mathcal{C}_X(U) = \{\varphi : U \to \mathbb{R} : \varphi \text{ continuous}\} \text{ for } U \subseteq X \text{ open}\,.$$

a) Give an example of a space X where $\mathcal{F}$ is not a sheaf.

b) Convince yourself that $\mathcal{C}_X$ is a sheaf.

c) Show that the sheafification of $\mathcal{F}$ is given by the sheaf $\mathcal{C}_X$.

d) Let $p : X \to \{\text{pt}\} =: Y$ be the constant map to a point. What is the sheaf $p_*\mathcal{C}_X$?

Note: Here we use sheafification for sheaves on an arbitrary topological space. The definition of sheafification here is just exactly the same as for schemes.

Solution.

a) For $X = \mathbb{R}$ with the usual topology consider the cover $U_i = \{(i, i+1)\}_{i \in \mathbb{R}}$ of X and the sections $f_i = (x \mapsto x) \in \mathcal{F}(U_i)$. Clearly all these functions are continuous and bounded on the intervals $(i, i+1)$ and they agree on overlaps. However, the unique function $X \to \mathbb{R}$ agreeing with f_i on U_i is the function $\mathbb{R} \to \mathbb{R}, x \mapsto x$, which however is not bounded and thus not an element of $\mathcal{F}(X)$. This is a violation of the sheaf axiom.

b) This follows since continuity is a local property.

c) There is a natural (inclusion) map $\rho : \mathcal{F} \to \mathcal{C}_X$ sending the bounded continuous function φ to the continuous function φ. Our first claim is that this map induces an isomorphism on stalks. Indeed, for $p \in X$ the inverse map $\mathcal{C}_{X,p} \to \mathcal{F}_p$ takes a gerbe $[(U, \varphi)]$ of a continuous function φ on a neighborhood U of p. After restricting to a smaller open set $U' \subseteq U$ (e.g. by taking $U' = \varphi^{-1}((\varphi(p) - 1, \varphi(p) + 1))$ the inverse image of a small interval around the value of φ at p) we can assume that φ is bounded on U', and thus that $[(U', \varphi|_{U'})] \in \mathcal{F}_p$. The map

$$\mathcal{C}_{X,p} \to \mathcal{F}_p, [(U, \varphi)] \mapsto [(U', \varphi|_{U'})]$$

is well-defined and an inverse to ρ_p.

Let $\overline{\rho} : \mathcal{F}^{\text{sh}} \to \mathcal{C}_X$ be the unique map such that ρ factors through the sheafification $\theta : \mathcal{F} \to \mathcal{F}^{\text{sh}}$ as $\rho = \overline{\rho} \circ \theta$. Then since both θ and ρ induce isomorphisms on the stalk level, also $\overline{\rho}_p$ is an isomorphism for all $p \in X$. Using Exercise 1 a) this proves the desired isomorphism $\overline{\rho} : \mathcal{F}^{\text{sh}} \xrightarrow{\sim} \mathcal{C}_X$.

d) A sheaf on a single-point space is always determined by its global sections (since there are only the two open sets $\emptyset, Y$ and a sheaf always has value $\{0\}$ on $\emptyset$). Plugging in the definition we see

$$(p_* \mathcal{C}_X)(Y) = \mathcal{C}_X(p^{-1}(Y)) = \mathcal{C}_X(X) = \{f : X \to \mathbb{R} : f \text{ continuous}\}.$$

Exercise 4. Let $n \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$. Prove that $\mathcal{O}_{\mathbb{P}^n}(d)^\vee \cong \mathcal{O}_{\mathbb{P}^n}(-d)$.
Note: For inspiration you can re-read the proof from [Gathmann, Example 13.23]. In your proof, you might want to use the presheaf $\mathcal{O}_{\mathbb{P}^n}(d)^{\vee,\text{pre}}$ given by $U \mapsto \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_{\mathbb{P}^n}(d)|_U, \mathcal{O}_U)$.

Solution. We have seen in class that for $U \subseteq \mathbb{P}^n$ open there exists a map

$$m_V : (\mathcal{O}_{\mathbb{P}^n}(-d))(V) \otimes_{\mathcal{O}_{\mathbb{P}^n}(V)} (\mathcal{O}_{\mathbb{P}^n}(d))(V) \to \mathcal{O}_{\mathbb{P}^n}(V)$$

of $\mathcal{O}_{\mathbb{P}^n}(V)$-modules, compatible with restriction maps to smaller open subsets. Let $\mathcal{O}_{\mathbb{P}^n}(d)^{\vee,\text{pre}}$ be the presheaf given by $U \mapsto \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_{\mathbb{P}^n}(d)|_U, \mathcal{O}_U)$.

We define a morphism of presheaves

$$\rho : \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n}(d)^{\vee,\text{pre}}.$$

Given $U \subseteq \mathbb{P}^n$ and $\psi \in \mathcal{O}_{\mathbb{P}^n}(-d)(U)$, the associated image under ρ is given by

$$\rho(\psi) : \mathcal{O}_{\mathbb{P}^n}(d)|_U \to \mathcal{O}_U, \varphi \in \mathcal{O}_{\mathbb{P}^n}(d)(V) \mapsto m_V(\psi|_V, \varphi) \in \mathcal{O}_U(V)$$

where $V \subseteq U$ is an open subset. The fact that the m_V are compatible under restriction shows that both $\rho(\psi)$ and then the map ρ itself give morphisms of presheaves of modules.

Let $\theta : \mathcal{O}_{\mathbb{P}^n}(d)^{\vee,\mathrm{pre}} \to \mathcal{O}_{\mathbb{P}^n}(d)^{\vee}$ be the morphism to the sheafification. Then we claim that $\theta \circ \rho$ is the desired isomorphism. To prove this we just observe that on the standard open cover U_i of $\mathbb{P}^n$ the sheaves $\mathcal{O}_{\mathbb{P}^n}(\pm d)$ are isomorphic to $\mathcal{O}_{U_i}$. Tracing through the isomorphisms and checking what $\theta \circ \rho$ does on the stalks, we find that it is just given by the map

$$\mathcal{O}_{\mathbb{P}^n,p} \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n,p}}(\mathcal{O}_{\mathbb{P}^n,p}, \mathcal{O}_{\mathbb{P}^n,p}), f \mapsto (g \mapsto fg)\,.$$

This is an isomorphism, so by Exercise 1 a) we have that $\theta \circ \rho$ is an isomorphism. Here we used that the stalk of the dual sheaf is the dual module of the stalk.